Announcements

- Office hrs by Yuve (TA) start today.
  Mondays 1-2 pm MATX 1101.
- My office hrs Wed 11-12:30 / Fri: 12:30-2 } in MATH 217.
- On ~~Wed~~ Wed, we'll have a guest lecture about python, computer project. Bring your computer!
- On Mon Feb 6, "basics quiz" 25 min, 2nd half of class.

  Covers everything up to the end of HW 3.
  vector spaces, bases, lin. dependence, ...
  + today's lecture.
- Homework due: Thursdays 10pm.
- Solutions to HW will be posted soon.

- ## Linear Transformations

Def: V, W - vector spaces over F.

$f: V \to W$ is called a linear transformation

if: $f(x+y) = f(x) + f(y)$ and

$f(\lambda \cdot x) = \lambda \cdot f(x)$ for $\lambda \in F$.

operations in V

operations in W

(another word: homomorphism of vector spaces.

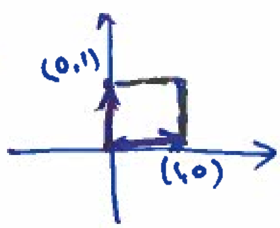
The set of all linear transformations from V to W is denoted by $\mathrm{Hom}(V, W)$

And $\mathrm{Hom}(V,W)$ is a vector space over F

(will put in homework)

Examples

①



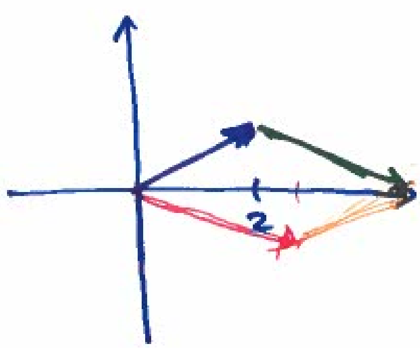
$f(x,y) = (2x + 3y,\ x - y)$

$f: \mathbb{R}^2 \to \mathbb{R}^2$

$\boxed{\mapsto \text{maps to}}$ tells us where the vector maps to

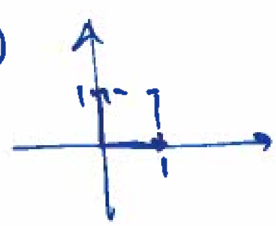
$(1,0) \mapsto (2\cdot 1, 1)$

$(0,1) \mapsto (2\cdot 0 + 3\cdot 1,\ 0-1)$
$= (3, -1)$

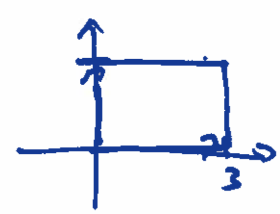


It is a lin. transf.
(exer: check it!)

②



$f(x,y) = (3x, 2y)$
linear transformation $\to$



Easier examples (1-dimensional)

0) a) $f: \mathbb{R} \to \mathbb{R}$

$f(x) = 3x$ – linear transf.

because: $3(x+y) = 3x+3y$
$3\cdot(\lambda x) = \lambda \cdot 3x$ in $\mathbb{R}$.

b) $f: \mathbb{R} \to \mathbb{R}$
$f(x) = x^2$

<u>Not</u> a linear transf.
e.g. because $(x+y)^2 \neq x^2+y^2$
$(\lambda \cdot x)^2 \neq \lambda \cdot x^2$

same for $f(x) = \frac{1}{x}$

$f(x) = e^x$ $\quad f(x) = \sin(x)$

<u>NOT</u> linear.

c) Trick question: $f: \mathbb{R} \to \mathbb{R}$ is a <u>linear function</u>: $f(x) = ax+b$ where $a, b \in \mathbb{R}$.

Then f is a <u>linear transformation</u> if and only if $b = 0$.

<u>Pf</u>: ($\Rightarrow$) if a lin. transf., want to prove: $b = 0$

apply it to $\lambda \cdot x$:

Need to have: $a\cdot(\lambda x) + b = \lambda\cdot(ax+b)$ for $\lambda \in \mathbb{R}$.

$$\lambda a x + b = \lambda a x + \lambda b$$

so $(\lambda - 1) b = 0$ for all $\lambda \in \mathbb{R}$

so $b = 0$.

($\Leftarrow$) $f(x) = a \cdot x$ → easy to check it is a lin. transf.

<u>Abstractly</u>: Let $A: V \to W$ be a linear transformation

~~Notation~~ Let $\{v_1, \dots, v_n\}$ be a basis of V.

Then A is <u>completely determined</u> by

the vectors $A(v_1), \dots, A(v_n) \in W$

and conversely, for <u>any</u> collection of vectors $w_1, \dots, w_n \in W$,

there exists <u>exactly one</u> linear transformation

$B: V \to W$ s.t. $B(v_1) = w_1, \dots, B(v_n) = w_n$.

the rest ~~and~~ is uniquely <u>determined</u>

<u>Proof</u>: Recall that $\{v_1, \dots, v_n\}$ is a basis (l.w.)

$\iff$ for every vector $v \in V$, there exists unique collection $c_1, \dots, c_n \in F$

s.t. $v = c_1 v_1 + \dots + c_n v_n$

So if $A(v_1) = w_1, \dots A(v_n) = w_n$,

then $A(v) = c_1 w_1 + \dots + c_n w_n$.

To do calculations in coordinates, we represent linear transformations with <u>matrices</u>

$A \rightsquigarrow$ <u>matrix</u> of A.

Matrices

Consider $V = F^n$, with the standard basis

$$e_1 = (1, 0 \ldots 0)$$
$$\vdots$$
$$e_n = (0, 0, \ldots 1)$$

$(x_i \in F)$

Any $v \in F^n$ has coordinates $v = (x_1, \ldots, x_n)$
$= x_1 \cdot e_1 + \ldots + x_n e_n$

Now let $A : F^n \to W$ be a linear transform.
it is determined by $w_1 = A(e_1), \ldots, w_n = A(e_n)$.

Think of $W = F^m$ with a standard basis.
will explain. write each w_i in coordinates in F^m.

Let $e_1' = (1, \underbrace{0 \ldots 0)}_{\text{length } m}$ $\leftarrow$ standard basis of $W = F^m$.
e_2'
$\vdots$
$e_m' = (0, 0, \ldots 1)$

coords of $w_n = A(v_n)$ in W.

$$\text{let } A = \begin{bmatrix} a_{11} & a_{12} & \cdot & a_{1n} \\ a_{21} & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

— matrix of the linear transf. A

coordinates of w_1 in F^m $= A(v_1)$

coords of $w_2 = A(v_2)$

This tells us how to get from a linear transf. to a matrix.

In our example $f(x,y) = (2x+3y,\ x-y)$.

matrix of this lin. transf.

is $A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$

$\begin{array}{cc} \uparrow & \uparrow \\ \text{image of } e_1 & \text{image of } e_2 \end{array}$

Recall: $(1,0) \mapsto (2,1)$
$(0,1) \mapsto (3,-1)$

How to go back from a matrix to a lin. transf.

look at the columns. call them $w_1, \dots, w_n$.

say: $A(e_1) = w_1, \dots,$ $A(e_n) = w_n$.

$A(\lambda_1 e_1 + \dots + \lambda_n e_n) = \lambda_1 w_1 + \dots + \lambda_n w_n$.

in practice, we do this:

$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 2x+3y \\ 1\cdot x - 1\cdot y \end{bmatrix} = (2x+3y,\ x-y)$$

coords of our vector

new vector

recovered our lin. transf

More about this on Friday.