

Office hours today: 11:30 - 1:30

- Gaussian elim. solving systems. ✓
- determinants
- eigenvalues  
↑ at the end

## Determinants

• "Example"  $2 \times 2$ -determinant.



$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ - a } 2 \times 2 \text{ - matrix}$$

$\det A \stackrel{\text{def}}{=} ad - bc$  - a scalar (happens to equal area of a parallelogram)  
- formula for  $\det A$  for  $2 \times 2$ -matrices

"det" is a map:  $n \times n$  square matrices over  $F \rightarrow F$

Let us talk about properties of this map for  $2 \times 2$ -matrices.

1) if you scale any row by  $\lambda$ ,  $\det(A)$  gets multiplied by  $\lambda$ :

useful!

$$\begin{bmatrix} \lambda a & \lambda b \\ c & d \end{bmatrix} \xrightarrow{\det} \lambda ad - \lambda bc = \lambda \det A.$$

2) if we fix  $c, d$ , replace  $a, b$  with  $a = a_1 + a_2$   
 $b = b_1 + b_2$

$$\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c & d \end{bmatrix} \xrightarrow{\det} (a_1 + a_2)d - (b_1 + b_2)c \\ = a_1 d - b_1 c + a_2 d - b_2 c \\ = \det \begin{bmatrix} a_1 & b_1 \\ c & d \end{bmatrix} + \det \begin{bmatrix} a_2 & b_2 \\ c & d \end{bmatrix}$$

(1) + (2) mean: "linear in each row" from last time.

3)  $\det \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$ . but also  $\det(A) = 0 \Leftrightarrow \text{rk}(A) < 2$

$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0 \Leftrightarrow ad - bc = 0 \Leftrightarrow ad = bc$   
suppose, say,  $c \neq 0, d \neq 0$   
Then  $\frac{a}{c} = \frac{b}{d}$   
Then rows are proportional.  $\Rightarrow \text{rk}(A)$

If  $c=0$  and  $d=0$ , we have a row of 0's  
so  $\text{rk}(A) \leq 1$

if  $c=0$   $d \neq 0$ , then:  $ad=0$  so  $a=0$   
then we have  $\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$  - a column of 0's  
so  $\text{rk}(A) \leq 1$ .

$\Leftarrow$ : a reverse this argument.

so:  $\det(A) \neq 0 \Leftrightarrow \text{rk}(A) = 2$   
 $\Leftrightarrow A^{-1}$  exists!

4)  $\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$ .

Now recall last time: we said, we want

a map  $D: n \times n$ -matrices over  $F \rightarrow F$

satisfying: (1) linear in each row (our (1) and (2) from the example)

(2)  $D(A) \neq 0 \Leftrightarrow \text{rk}(A) = n$

(3)  $D(\text{Id}) = 1$ .

"pins down" the const.  $c$

$\uparrow$  requires proof

The maps satisfying these two properties form a 1-dim. vector space

if  $D_1, D_2$  are such maps, then  $D_1 = cD_2$  for some  $c \in F$ .

Together, these properties guarantee that there is at most one such map  $D$ .

We skip the proof that  $\dim(\text{space of maps satisfying (1) and (2)}) \leq 1$   
 (it is in 6.1 (recommended reading))

We will build the det map. (which shows it exists!)

Start with 1x1-matrices:

1) (1x1-matrices give linear maps:  $F \rightarrow F$ )  
 a scalar:  $[a] \leftarrow$  1x1-matrix.  
 $a \in F$

$$\det([a]) = a.$$

$$\det([a]) = 0 \iff a = 0.$$

2) 2x2-matrices: we just did it.

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \cdot \det[d] - b \det[c] = ad - bc$$

take an entry in the first row, and remove the column containing it

3) 3x3-matrices

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

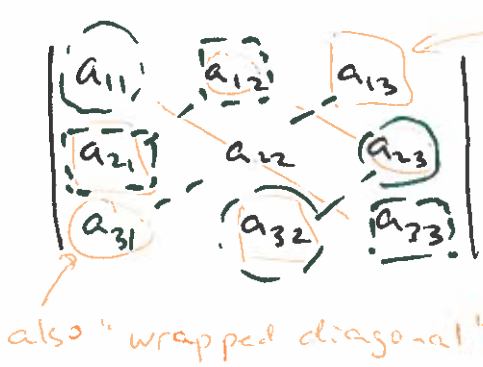
black block

signs alternate!

↑ blocks are "minors"

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Check that it agrees with:



$$= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33}$$



each term has exactly one entry from each row and each column!

(this leads to an alternate description of 'det' map, signs are hard to describe.)  
+ or - ?

For  $n \times n$ :

$$\det \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \hline & & & \end{vmatrix} = a_{11} \det A_{11} - a_{12} A_{12} + a_{13} A_{13} - \dots \pm a_{1n} A_{1n}$$

↑  
minor of  $a_{11}$ :  
obtained by throwing away row and column containing it

## Key point:

- it doesn't change with elem. row / col ops of the form  $R_i \rightarrow R_i + \lambda R_j$
- when you multiply a row or column by a scalar, det multiplies by that scalar.
- when you swap two rows or columns, det changes sign.

("easy" to derive from the recursive def'n.)  
easy from the axioms.

So: we can do row ops to get a simpler matrix and get det that way.

Read: 6.1, 6.2