Today: More review, quotient spaces.

- $Ax = b$ – system of lin. equations

has → no solutions
→ one solution
→ infinitely many solutions

How does this relate to properties of A?

- If A is full rank, i.e. $\mathrm{rk}(A) = m$, then it has solutions for every b.
  $A: F^n \to F^m$

  If $\mathrm{rk}(A) < m$, then has solution(s) for some b ~~and~~ but not for all b.

- If $\ker(A) = \{0\}$, then (at most) one solution for every b.

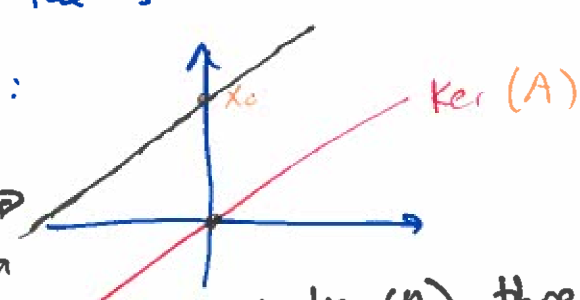
Generally,

ALL solutions are of the form $x_0 + u$, where x_0 is any solution $Ax_0 = b$

and $u \in \ker(A)$

When there is more than one solution to a linear system, then the solutions form a "translated subspace":

Given:
$Ax_0 = b$
Find all the solutions to $Ax = b$.



line parallel to ker(A) through x_0

When looking for $\ker(A)$, we solve:

$$Ax = 0 \qquad (\text{take } b = 0)$$ ← do not need to augment the matrix

Now if we change the RHS: put in some b,

$Ax = b$, the elem. row operations you have to do are <u>the same</u> for all b.

- If A is a square matrix (i.e., $A: F^n \to F^n$)

  then $\ker(A) = \{0\} \iff rk(A) = n$

  solutions exist for all b $\iff$ there is <u>unique</u> solution for all b.

$\iff \det(A) \neq 0$

$\iff A^{-1}$ exists

And the solutions can be written as:

$$Ax = b \iff \boxed{x = A^{-1} b.}$$

- Inconvenient way to solve equations and find A^{-1}:

  . Cramer's rule:

$$x_i = \frac{\det \begin{pmatrix} a_{11} & \boxed{} & a_{1n} \\ a_{n1} & & a_{nn} \end{pmatrix}}{\det(A)}$$

ith column replaced by b

  . for A^{-1}:

$$A_{ij} = \begin{pmatrix} & | & | & \\ \hline & & a_{ij} & \\ \hline & | & | & \end{pmatrix}$$

remove ith row, jth column

← $(n-1)\times(n-1)$ matrix

$$A^{-1} = \frac{\left((-1)^{i+j} \det(A_{ij})\right)^t_{\substack{i=1,\dots,n \\ j=1,\dots,n}}}{\det(A)}$$

$n \times n$ matrix

then we divide every entry by $\det(A)$.

What it does for $\underline{2\times 2}$-matrix:

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

Question: when solving a system, how to name the parameters:

suppose after row reductions, we got this:

column has no pivot

x_3 is a free variable

$$\begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & \boxed{2} & 0 & b_1 \\ 0 & 1 & \boxed{0} & 0 & b_2 \\ 0 & 0 & \boxed{0} & 1 & b_3 \\ 0 & 0 & \boxed{0} & 0 & 0 \end{array}\right] \\ \qquad\quad x_3 \end{array}$$

This gives:
$$\begin{cases} x_1 + 2x_3 = b_1 \\ x_2 = b_2 \\ x_4 = b_3 \end{cases}$$

$$(*) \begin{cases} x_1 = b_1 - 2x_3 \\ x_2 = b_2 \\ x_4 = b_3 \end{cases}$$

Remember: looking for a <u>column vector</u> $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

this says, our vector is: $\underbrace{\begin{bmatrix} b_1 \\ b_2 \\ 0 \\ b_3 \end{bmatrix} + x_3 \cdot \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\text{basis of } \ker(A)}}_{\text{translation of } \ker(A) \text{ by this vector}}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ plug in $(*)$ for the variables

- If you were asked about the basis of Im(A): look at the pivot columns, take the original columns of A (in our case, columns 1, 2, 4 of A give basis of Im(A))

<u>Question</u> about $\det(A^{-1} A^t B A)$

$A = \begin{bmatrix} - & - \\ - & - \end{bmatrix}$ $B = \begin{bmatrix} - \\ - \end{bmatrix}$ (two 4×4 matrices)

Key: simplify first:
recall $\det(AB) = \det(A)\det(B)$

and $\det(A^{-1}) = \dfrac{1}{\det(A)}$

[exists iff $\det(A) \neq 0$]
"if and only if"

So this expression is:

$\cancel{\det(A)}^{-1} \cdot \det(A^t) \det(B) \cdot \det(A)$

$\det(A^t) = \cancel{\det(A)}$

Need $\det(A)$, $\det(B)$

start with $\det(B)$.
compute $\det(A)$ (if $\det(B) \neq 0$ otherwise, do not need it)

Quotient vector spaces (Not on the exam, but is important for mathematicians ☺)

- Equivalence relation "a way to identify elements of a set":

read formal def'n in 4.4

example: recall our field of p elements:

$\mathbb{F}_p = \{0, 1, \ldots, p-1\}$ operations "mod p": do addition, mult. but then take remainder when divided by p.

Now: I'll say two integers are equivalent if they have the same remainder when divided by p.

So, 1 and $p+1$, $2p+1$, $3p+1, \ldots$ are all equivalent.

"having the same remainder when divided by p" is an equivalence relation on $\mathbb{Z}$.

Elements that are equivalent to each other form an equivalence class

So our field $\mathbb{F}_p$ is in fact the set of equivalence classes of integers.

Notation: $[x]$ = equivalence class of x
$= \{y : x \text{ is equivalent to } y\}$

"$x \sim y$"

our equivalence

$$[3]_p = \{\ldots, 3-p, 3, 3+p, 3+2p, 3+3p, \ldots\}$$

subset of $\mathbb{Z}$

Make a new set : the set of equivalence classes modulo p.

$$= \{[0]_p, [1]_p, \ldots, [p-1]_p\}$$

Def Let V be a vector space.
and $W \subset V$ be a subspace.

Say $x \sim y$ if $x - y \in W$

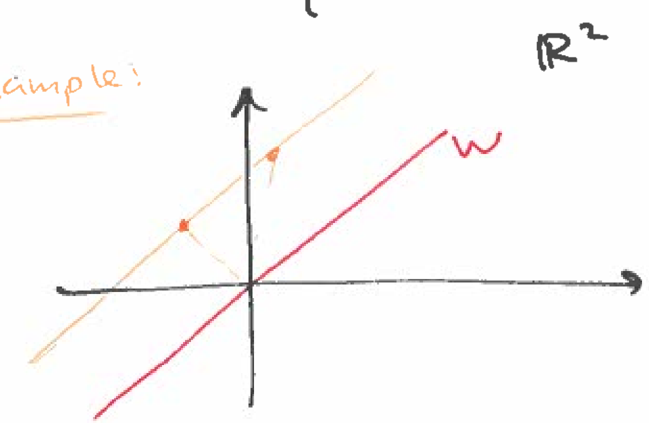
This defines an equiv. relation.

And we can define a new vector space :

V/W "quotient space"

$= \{$ set of equivalence classes by this relation $\}$

Example:



equivalence classes are lines parallel to W

$$\boxed{\mathrm{Im}(A) \cong V/\mathrm{Ker}(A)}$$

Another way of saying that all solutions to $Ax = b$ $\in \mathrm{Im}(A)$ are of the form $x_0 + u$ where $u \in \mathrm{Ker}(A)$ and $Ax_0 = b$.