Euclidean spaces

- Extra structure on a vector space V over ~~a field~~ $\mathbb{R}$:
  - dot product (or scalar product)

~~vector / addition~~

Abstract definition The dot product is a function ~~from~~ $(,) : V \times V \to \mathbb{R}$

satisfying:

1) bilinear: in $\mathbb{R}$ ← (x, y) is linear in x if y is fixed
   in V — linear in y if x is fixed

   This means:
   $$(x_1 + x_2, y) = (x_1, y) + (x_2, y)$$
   $$(\lambda x, y) = \lambda(x, y) \quad \text{for } \lambda \in \mathbb{R}$$
   $$(x, y_1 + y_2) = (x, y_1) + (x, y_2)$$
   $$(x, \lambda y) = \lambda(x, y) \quad \text{for } \lambda \in \mathbb{R}.$$

2) symmetric: $(x, y) = (y, x)$

3) positive-definite: $(x, x) \geq 0$ and $(x, x) > 0$ if $x \neq 0$.

   this is why we only work over $\mathbb{R}$.

Another term for a dot product:

"symmetric bilinear positive-definite form"

function of 2 variables, linear in each.

Example: $x, y \in \mathbb{R}^n$

Define $(x, y) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$

- the usual dot product $x \cdot y$

In fact, there exist infinitely many different scalar products (symmetric bilinear pos. def. forms) on $\mathbb{R}^n$ (or on any real vector space).

<u>Example</u>: $A: \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear op.,

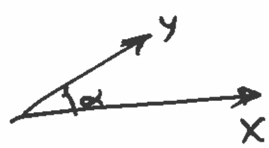
then $(x,y)_{new} = (Ax, Ay)$ - gives us a new scalar product.

<u>Why we care</u>: this gives us the notion of
- distance (or length), and
- angles.

<u>Def</u>: $\|x\| = \sqrt{(x,x)}$

"the <u>norm</u> of x"

We could define the angle between x and y via its cosine:



$$x \cdot y = \|x\| \|y\| \cos\alpha$$

so we define $\alpha = \cos^{-1}\left(\dfrac{(x,y)}{\sqrt{(x,x)}\sqrt{(y,y)}}\right)$

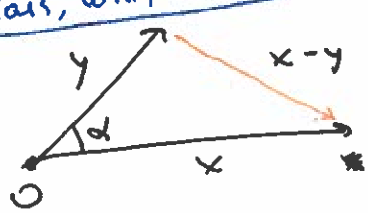
<u>Something to prove before we can do it</u>:

$$\left|\frac{(x,y)}{\sqrt{(x,x)}\sqrt{(y,y)}}\right| \leq 1. \quad (\text{in order to define } \cos^{-1}(\ldots))$$

next class, will prove it for the general dot product.

Let's compute $(x-y, x-y) = \|x-y\|^2 \geq$

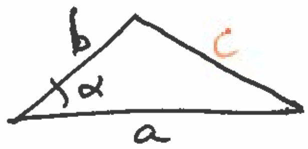
$$= (x,x) - (y,x) - (x,y) + (y,y)$$



Get: $\boxed{\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2(x,y}$

For the <u>standard</u> dot product, it agrees with the usual angles.

Law of cosines:



$$\boxed{c^2 = \underbrace{a^2}_{\|x\|^2} + \underbrace{b^2}_{\|y\|^2} - 2ab\cos\alpha}$$

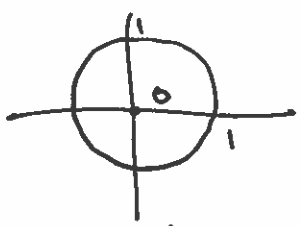
So $(x,y) = \|x\|\,\|y\|\cos\alpha$.

We just proved that with the usual notion of length, the angle agrees with the formula we get from the dot product.

Example: Now that we have the norm, we have unit circle: $\{x \in \mathbb{R}^2 : \|x\| = 1\}$.

(in $\mathbb{R}^2$)

The usual dot product:



(our familiar unit circle)

New dot product:

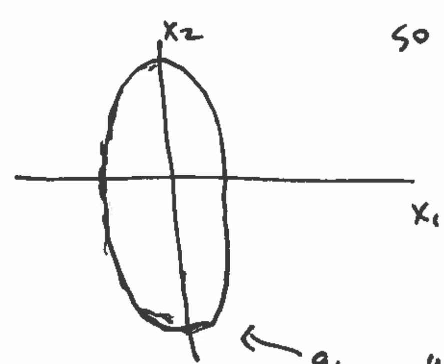
Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

stretches by a factor of 2 along x.

So $\|x\|_{new} = \sqrt{(Ax, Ax)} = \sqrt{4x_1^2 + x_2^2}$

$x = (x_1, x_2)$

If we use the new dot product to define angles, they will be different from our usual angles.



an ellipse

so the unit circle is:

$4x_1^2 + x_2^2 = 1$

in the standard coords.

What we really care about, is the notion of orthogonality: x and y are orthogonal with respect to $(\,,\,)$ if $\boxed{(x,y) = 0.}$

<u>Example</u> of dot product in an infinite-dim. vector space

- consider the space of all continuous functions
  $f : [-1,1] \to \mathbb{R}$

They form a real vector space. (infinite-dim'l)

Define $$(f,g) = \int_{-1}^{1} f(x)\, g(x)\, dx .$$

<u>Claim</u> This is a symmetric bilinear positive-definite form.

let's check linearity in the first argument:

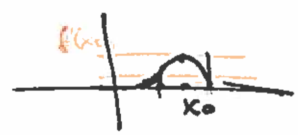
$$(f_1 + f_2 , g) = \int_{-1}^{1} (f_1 + f_2)(x)\, g(x)\, dx$$

$$= \int_{-1}^{1} f_1(x)\, g(x)\, dx + \int_{-1}^{1} f_2(x)\, g(x)\, dx .$$

Similarly,

$$(\lambda f, g) = \lambda (f,g).$$

Positive-definite: $$(f,f) = \int_{-1}^{1} f^2(x)\, dx \geq 0 .$$



why $(f,f) \neq 0$ for $f \neq 0$:

since f is continuous, if $f \neq 0$ then exists $x_0 \in [-1,1]$ s.t. $f(x_0) \neq 0$ so $f^2(x_0) > 0$. Then $f^2 > 0$ on some neighbourhood of x_0 since f is continuous. The integral over this neighbourhood will be positive.

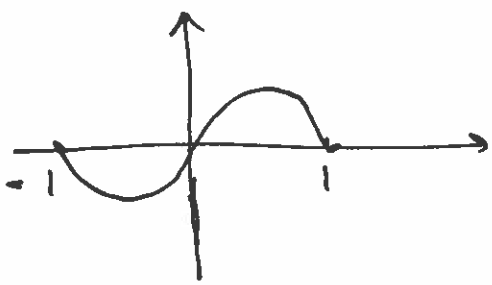
Thus we can define the "L^2-norm" of f:

$$\|f\|_2 = \sqrt{\int_{-1}^{1} f^2(x)\, dx}$$

The functions f, g are "orthogonal" if

$$(f,g) = 0 \quad \text{, i.e.} \quad \int_{-1}^{1} f(x)\, g(x)\, dx = 0.$$

Example: $f(x) = \sin(\pi x)$, $g(x) = 1$ — orthogonal.

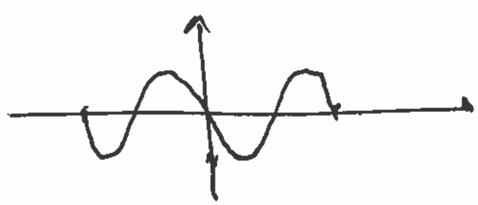


$\sin(\pi x)$, $\cos(\pi x)$ also orthogonal, but harder:

$$\int_{-1}^{1} \sin(\pi x) \cos(\pi x) = $$

use double angle formula

$$= \frac{1}{2} \int_{-1}^{1} \sin(2\pi x) = 0.$$



A vector space V over $\mathbb{R}$ with an inner product (dot product) (scalar product)

is called a Euclidean space.

So, the space of cont. fns : $[-1,1] \to \mathbb{R}$ is a Euclidean space.

Def: A basis $e_1, \dots, e_n$ in a Euclidean space V is called orthonormal if $(e_i, e_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

(all orthogonal, $\underbrace{\text{length } 1}_{\text{"normal"}}$)

"ortho"

Magic: If $e_1, \dots, e_n$ is an orthonormal basis then if $x = \lambda_1 e_1 + \dots + \lambda_n e_n$, you can find the λ's using the dot product:

$$\boxed{\lambda_i = (x, e_i)}$$