Today: · Euclidean spaces: V - vector space
$(\, , \,)$ - inner product.

The upshot: the only kind of basis you want to work with in a Euclidean space is an orthonormal basis:

that is, a basis $\{w_1, \dots, w_n\}$ such that

$$(w_i, w_j) = \delta_{ij} \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

"delta-i-j"
(the "Kronecker δ")

Aside:
Example (of the δ-notation) ← an important family of linear functionals of a space of functions

Let V = the space of all continuous functions $f: \mathbb{R} \to \mathbb{R}$

Define a linear operator $A_0: V \to \mathbb{R}$ by

$$A_0(f) = f(0)$$

More generally, fix $x_0 \in \mathbb{R}$, define $A_{x_0}(f) = f(x_0)$

- A linear operator from V to a 1-dimensional vector space is called a linear functional

And these linear functionals we just defined have a standard name: they are called "the δ-function":

Standard notation:
$$\boxed{\begin{aligned} \delta_0(f) &= f(0) \\ \delta_{x_0}(f) &= f(x_0) \end{aligned}}$$

$\mathrm{Ker}(\delta_{x_0}) = \{ f: \mathbb{R} \to \mathbb{R} \text{ cont. s.t. } f(x_0) = 0 \}$ - infinite-dim.

Back to orthonormal bases

$$(w_i, w_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

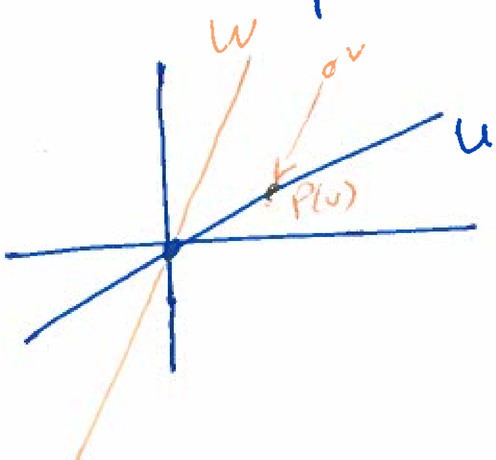
vectors are mutually orthogonal and have length 1.

- Normally, when we talk about a V - vect. space, we say let $v_1, \dots, v_n$ be a basis, ...
- In a Euclidean space, we say, let $v_1, \dots, v_n$ be an orthonormal basis --

Why do they exist?

First, projections.

Recall: $V = U \oplus W$, we have a projection operator $P_u : V \to U$ - projection onto U along W



$V = V \oplus W$ means, every v has unique decomp.

$v = u + w$

$u \in U$, $w \in W$

$P(v) := u$.

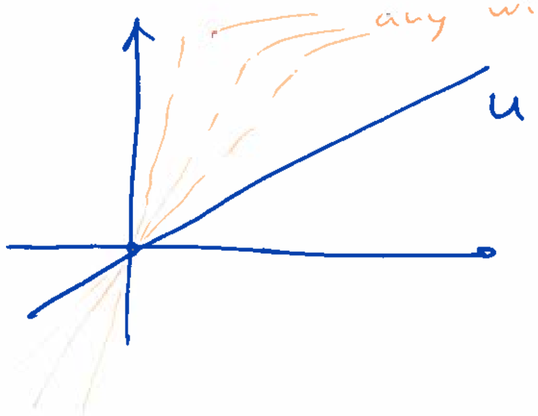
- This operator has the property $P^2 = P$.

In fact, any linear operator satisfying this property is a projector:

Let P satisfy $P^2 = P$. Then Take $U = \mathrm{Im}(P)$
$W = \mathrm{Ker}(P)$.

We proved in h/w that then $V = U \oplus W$.

• In an arbitrary vector space, given a subspace U, there are infinitely many (unless your field is finite) possible projections onto U along various spaces W:



(there are many W s.t. $V = U \oplus W$).

In a Euclidean space, there is <u>the best</u> one of them, called the <u>orthogonal complement</u> of U.

(Recall)

<u>Def</u> $U^\perp = \{ v \in V : (u,v) = 0 \quad \forall u \in U \}$.

<u>Thm:</u> $U \oplus U^\perp = V$.

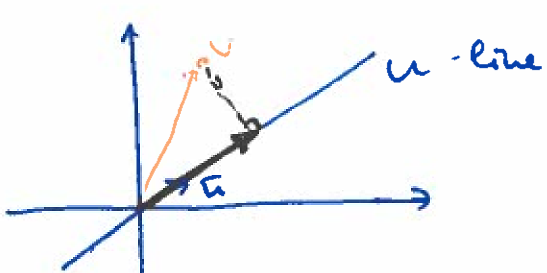
<u>Pf</u>: <u>Recall</u>: direct sum means: every $v \in V$ <u>has</u> a decomposition $v = u + w$, $u \in U$, $w \in U^\perp$ <u>and</u> such a decomp. is <u>unique</u> (which is equivalent to saying $U \cap U^\perp = \{0\}$)

existence of this decomp. says: $U + U^\perp = V$ or $L(U, U^\perp) = V$.

• <u>Easy part</u>: $U \cap U^\perp = \{0\}$: suppose $v \in U \cap U^\perp$. Then $(v,v) = 0$ Then $v = 0$.

• So: existence of this decomposition:

$v \in V$.



$\mathrm{Proj}_U v = \vec{u}_0 \cdot (v, u_0)$.

unit basis vector of our line U.

Why this works for a line?

$v - u_0 \cdot \underbrace{(v, u_0)}_{\text{scalar}} \perp u_0$ ← need to show this.

$(u_0, v - u_0 \cdot (v, u_0)) = (u_0, v) - \underbrace{(u_0, u_0)}_{=1} \cdot (v, u_0)$

because u_0 is a ~~the~~ <u>unit</u> vector

$= 0.$

So: when projecting onto a line,

the component <u>along</u> the line is

$u = (v, u_0) \cdot u_0$, where u_0 is a unit basis vector

$w = v - (v, u_0) \cdot u_0$ is the <u>orthogonal</u> component.

So we proved our theorem when U is <u>a line</u>.

To do it in general, we ~~[illegible]~~ need an <u>orthonormal basis</u> for U. Then we'll have a formula for projection onto U.

<u>Lemma</u> Suppose we had an <u>orthonormal</u> basis $u_1, \dots, u_k$ in U.

Then $\operatorname{Proj}_U v = \sum_{i=1}^{k} (v, u_i) u_i$

↗ <u>orthogonal</u> projection onto U

$$= \underbrace{\underbrace{(v, u_1)}_{\text{comp. of } v \text{ along } u_1} u_1 + \dots + \underbrace{(v, u_k)}_{\text{comp. of } v \text{ along } u_k} u_k}_{\in U}$$

To prove: $v - \operatorname{Proj}_U v$ is orthogonal to U.

We compute: $\left(v - \sum_{i=1}^{k} (v, u_i) u_i, \underset{\in U}{u}\right) =$

$$= (v, u) - \sum_{i=1}^{k} (v, u_i)(u_i, u)$$

write $u = \sum_{i=1}^{k} a_i u_i$ - decomposition of u into the components along u_i in U.

$$= \sum_{i=1}^{k} a_i (v, u_i) - \sum_{i=1}^{k} (v, u_i)\underbrace{(u_i, u)}_{= a_i} = 0.$$

why $(u, u_i) = a_i$

Plug in $u = \sum_{i=1}^{k} a_i u_i$:

$$(u, u_j) = \left(\sum_{i=1}^{k} a_i u_i, u_j\right)$$

$$= \sum_{i=1}^{k} a_i \underbrace{(u_i, u_j)}_{= \delta_{ij}} = a_j \cdot 1$$

This says: all terms are 0 except for the j-th term

The j-th term is $a_j \cdot 1$

Gram - Schmidt orthogonalization process

Given a basis $\{v_1, \dots, v_n\}$ in V, how to make it into an orthonormal basis?

- Do it inductively:

1) • make v_1 a unit vector by replacing it with

$$\tilde{v}_1 = \frac{v_1}{\|v_1\|}.$$

2) Replace v_2 with $\tilde{v}_2' = v_2 - \mathrm{Proj}_{\tilde{v}_1} v_2 = (v_2, \tilde{v}_1)\tilde{v}_1.$

$$= \underbrace{v_2 - (v_2, \tilde{v}_1)\tilde{v}_1}_{\neq 0 \text{ b/c } v_1, v_2 \text{ were}}$$

So: $\tilde{v}_2' \perp \tilde{v}_1$.

3) Make $\tilde{v}_2'$ into a unit vector:

$$\tilde{v}_2 = \frac{\tilde{v}_2'}{\|\tilde{v}_2'\|} = \frac{v_2 - (v_2, \tilde{v}_1)\tilde{v}_1}{\|v_2 - (v_2, \tilde{v}_1)\tilde{v}_1\|}$$

4) Let $\tilde{v}_3' = v_3 - \mathrm{Proj}_{L(\tilde{v}_1, \tilde{v}_2)} v_3$

$$= v_3 - \underbrace{\left((v_3, \tilde{v}_1)\cdot\tilde{v}_1 + (v_3, \tilde{v}_2)\cdot\tilde{v}_2\right)}_{\text{not } 0 \text{ b/c}}$$

v_1, v_2, v_3 are lin. indep.

Let $\tilde{v}_3 = \dfrac{\tilde{v}_3'}{\|\tilde{v}_3'\|}$

...

This way we build an orthonormal basis, and it finishes the proof of the theorem.

Next time: Example of how to use it; Sections 8.3 and a bit of 8.4