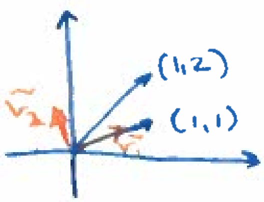
Last time

- Gram - Schmidt orthogonalization process

$v_1, \dots, v_n$ – a basis of a vector space V with Euclidean inner product $(\, , \,)$ $\rightsquigarrow$ $\tilde{v}_1, \dots \tilde{v}_n$ – an orthonormal basis: $\|\tilde{v}_i\| = 1$ and $(v_i, v_j) = \delta_{ij}$

Example:



$v_1 = (1,1)$
$v_2 = (1,2)$

the usual dot product:

Process: 1) make v_1 unit: $\tilde{v}_1 = \frac{v_1}{\|v_1\|} = \boxed{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}$

2) $\tilde{v}_2{}' = v_2 - \mathrm{proj}_{\tilde{v}_1} v_2 = (1,2) - \tilde{v}_1 \cdot (v_2, \tilde{v}_1)$

$= (1,2) - \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \cdot \frac{3}{\sqrt{2}} = (1,2) - \left(\frac{3}{2}, \frac{3}{2}\right)$

$= \left(-\frac{1}{2}, \frac{1}{2}\right)$

right direction but not unit yet

3) make $\tilde{v}_2{}'$ unit:

$\tilde{v}_2 = \frac{\tilde{v}_2{}'}{\|\tilde{v}_2{}'\|} = \boxed{\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}$

- From this, we proved that in a Euclidean space V, if U is a linear subspace, $V = U \oplus U^{\perp}$

orthogonal complement $= \{v \in V : (v, u) = 0 \ \forall u \in U\}$

- Now we can go back to justify: row rank = column rank.

- A real finite-dim. vector space can be made into a Euclidean space (using dot product in $\mathbb{R}^n$).

The point: the space spanned by the columns of A = the image of the linear operator A.

$A : V \to V$

$\updownarrow$

matrix A w.r.t. our chosen basis.

Think of V as $\mathbb{R}^n$ with the standard basis.

With respect to the dot product, then the space spanned by the rows of A is $\operatorname{Ker}(A)^\perp$

Important note: $\operatorname{Im}(A)$, $\ker(A)$ do not depend on the choice of basis in V.

The matrix for A does.

The notion of "$\perp$" also depends on the choice of (,)

So: this works when the choice of basis agrees with (,) (namely, is orthonormal).

Why does this work:

$$\begin{array}{c} r_1 \\ \\ r_n \end{array} \begin{pmatrix} a_{11} & - & - & a_{1n} \\ & & & \\ a_{n1} & - & - & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \underset{\substack{\text{def} \\ \text{of } A\cdot x}}{=} \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ - \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{pmatrix}$$

row vectors

$$= \begin{pmatrix} r_1 \cdot x \\ \vdots \\ r_n \cdot x \end{pmatrix}$$

using the dot product (this means, we are using the "standard basis" to write the matrix)

Now: $x \in \operatorname{Ker} A \iff Ax = 0$

$\iff \begin{cases} r_1 \cdot x = 0 \\ \vdots \\ r_n \cdot x = 0 \end{cases} \iff x \in (\text{row space})^\perp$, row space $= L(r_1 \ldots r_n)$

So we proved: $\boxed{\operatorname{Ker} A = L(r_1 \ldots r_n)^\perp}$

Exer: $(U^\perp)^\perp = U$.

Then $(\ker A)^\perp = \left(L(r_1, \dots, r_n)^\perp\right)^\perp = L(r_1, \dots, r_n)$.

Then we have: row space $= \ker(A)^\perp$

column space $= \mathrm{Im}(A)$.

$V = \ker(A) \oplus \ker(A)^\perp$ ← our theorem applied to $U = \ker(A)$

Then $\boxed{\dim\left((\ker A)^\perp\right) = \dim(V) - \underbrace{\dim(\ker(A))}_{\text{"nullity" of } A}}$

but we also have $\mathrm{rk}(A) = \dim(\text{column space})$

$= \dim(\mathrm{Im}(A))$

and: dimension formula

$\dim V = \dim(\mathrm{Im}\, A) + \dim(\ker(A))$

so $\boxed{\text{column rank} = \dim V - \dim(\ker(A))}$

More about matrices and Euclidean spaces:

Orthogonal matrices and isometries (§8.3).

Def: Let V, W be Euclidean spaces (with inner products $(\,,\,)_V$ and $(\,,\,)_W$ respectively.)

An isometry between V and W is a linear operator $A: V \to W$ s.t.

$$(Av_1, Av_2)_W = (v_1, v_2)_V$$

for all $v_1, v_2 \in V$.

"preserves metric"

distances

Exer: A is an isometry $\iff$ $\forall v \in V$
$\|v\|_V = \|Av\|_W$

hint: think of $(v+w, v+w) = \|v+w\|^2$

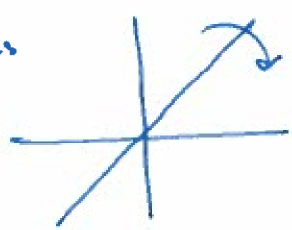
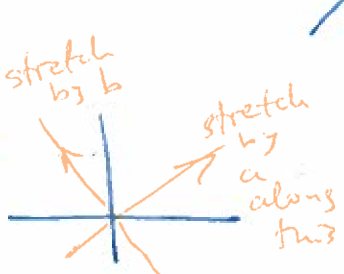
Def: A square matrix A is called orthogonal if $AA^t = Id$

Proposition let V be a Euclidean space
let $\{v_1, \dots, v_n\}$ be an orthonormal basis in V
Then $A: V \to V$ is an isometry $\iff$
The matrix of A with respect to this basis
is orthogonal.

Pf: exer. (see the book)

Example isometries in $\mathbb{R}^2$:
usual dot product

recall: bijective linear transf : $\mathbb{R}^2 \to \mathbb{R}^2$

- Id – an isometry.
- rotations  – isometries
- reflections  – isometries
- 

dilations – not isometries
- shear transf. ← see the book (or not) – not isometries

- We will be able to classify all lin. transf. : $\mathbb{R}^2 \to \mathbb{R}^2$
  (and even generally : $F^n \to F^n$) ← without complete proof

$A : V \to V$

Prop: A is orthogonal (i.e. an isometry) $\Rightarrow$ A is bijective ($\ker(A) = \{0\}$ and $rk(A) = \dim V$)

Pf: $(Av, Av) = (v, v)$

So $Av = 0 \Rightarrow (Av, Av) = 0 \Rightarrow (v, v) = 0$
$\Rightarrow v = 0$.

So $\ker(A) = \{0\}$. Then $rk(A) = \dim V$.

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§8.4 - optional reading

Next class: 9.1, 9.2 Eigenvalues.