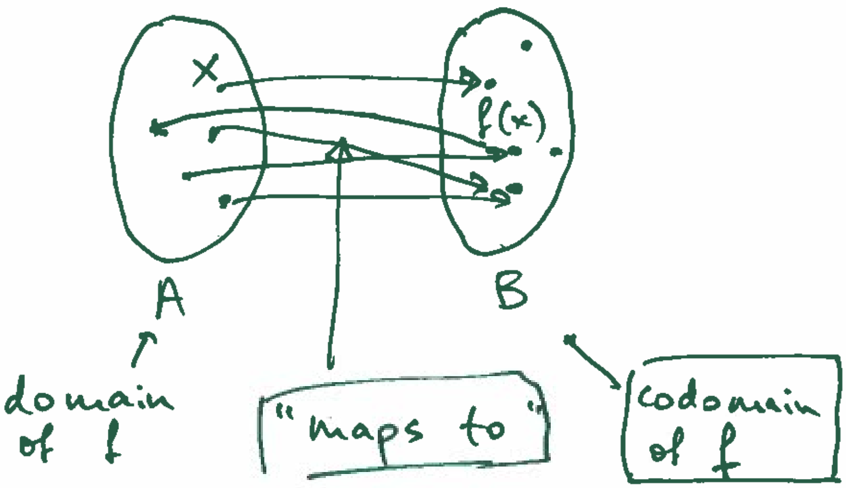
Today: <u>functions</u>

$f: A \to B$



<u>Function</u>

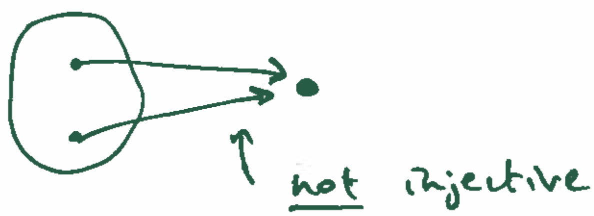
for each element of A there has to be <u>exactly one</u> outgoing arrow

$x \mapsto f(x)$

$f(x)$ is called the <u>image</u> of x.

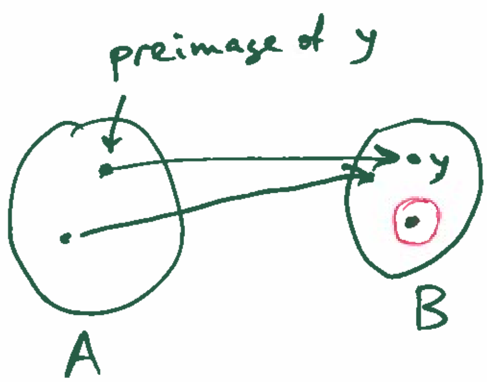
<u>Def</u>: ① $f: A \to B$ is called <u>injective</u> if

$$x \neq y \Rightarrow f(x) \neq f(y)$$



"distinct elements of A have distinct images"

② A function $f: A \to B$ is called <u>surjective</u> if every element of B has a <u>preimage</u> (inverse image) in A:



← no "lonely" dots in B.

the usage of if in definitions is different from logic!

definition.

Math 223, Diagnostic quiz.

Let A be a set of red dots, and B be a set of blue dots. We call a function $f : A \to B$ *nice* if for every blue dot there is a red dot that maps to it: for every blue dot $b \in B$ there is a red dot $a \in A$ such that $f(a) = b$.

1. Draw a picture illustrating the concept of a nice function.
2. Let us call a blue dot *lonely* if there is no red dot that maps to it. State the converse, contrapositive, and negation of the following statement:

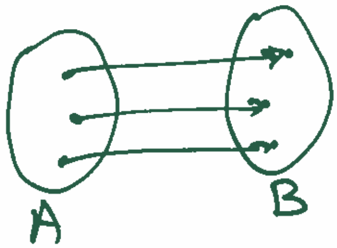
   If a function is nice, then no blue dot is lonely.

   Label each of your statements as True/False (no proof needed, but do not guess).

In definitions "if" means "if and only if".

Because of this, "function is nice" $\Updownarrow$ "no blue dot is lonely" are equivalent.

A function is called bijective if it is both surjective and injective



Composition of functions

Def: given $f: A \to B$, $g: B \to C$

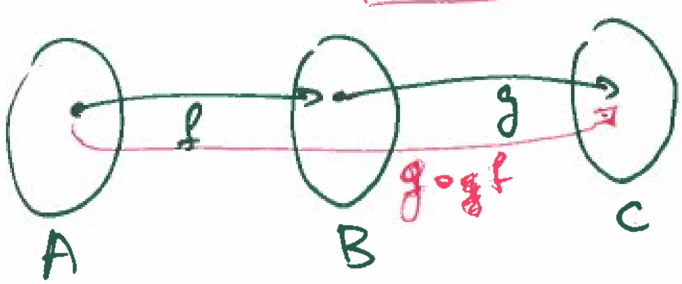
define $g \circ f : A \to C$

\circ

$(g \circ f)(x) = g(f(x))$ for $x \in A$.

Domain: A

codomain: C



Remark: Def.: for $f: A \to B$ the range of f is $f(A) = \{ y \in B : \exists x \in A, f(x) = y \}$

– the image of the set A.

Also, for $y \in B$, the inverse image (preimage) of y

$f^{-1}(y) = \{ x \in A : f(x) = y \}$ – a subset of A.

do not confuse with inverse function!

$\underline{\text{Exer}}$: $\underbrace{f(x)}_{f: A \to B}$ is surjective $\iff$ $\forall\, y \in B$
$f^{-1}(y) \neq \emptyset$.

$f(x)$ is bijective $\iff$ $\forall\, y \in B$
there is exactly one element in $f^{-1}(y)$.

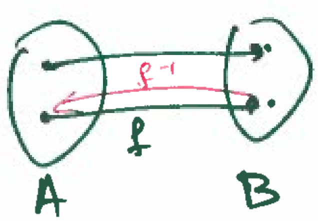
$\underline{\text{Exer}}$: Someone was asked to prove that a certain $f: A \to B$ is surjective.
They say: " let $\cancel{f(x) \in} B$ . . . .
already assuming what you want to prove.

Correct way: let $y \in B$.
... then prove there is an $x \in A$
s.t. $f(x) = y$.

Inverse functions

1. Identity function.
for any set A, there is the identity function
$id_A : A \to A$, $id_A(x) = x$.

2. If $f: A \to B$ is a bijective function
then we can define the inverse function



$f^{-1}: B \to A$.
$f^{-1}(y) =$ the $x \in A$ s.t. $f(x) = y$
makes sense b/c f is bijective

f^{-1} is a well-defined function <u>only</u> when f is bijective.

If $f: A \to B$ is not bijective, but you <u>want</u> an inverse, it might help to change A, B.

<u>Examples</u>: arcsin, arctan

$\sin(x): \mathbb{R} \to \mathbb{R}$ is not injective, not surjective.

<u>restrict</u> it to $[0, 2\pi)$

restrict the codomain to $[-1, 1]$

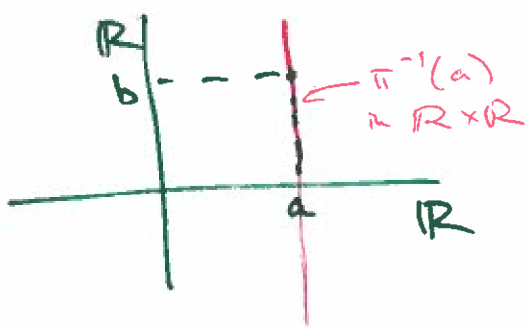
$\sin(x): [0, 2\pi) \longrightarrow [-1, 1]$ is bijective

Now the inverse exists.

<u>Example</u>: projections.

<u>Recall</u>: Cartesian (direct) product of sets:

$$A \times B = \{(a,b) \mid a \in A,\ b \in B\}.$$



<u>projections</u>

$$\pi_1: A \times B \to A$$
$$(a,b) \mapsto a$$

$$\pi_2: A \times B \to B$$
$$(a,b) \mapsto b$$

<u>Exer</u>: $\pi_1^{-1}(a) = \{ \cancel{(a,b)}\ (x,y) \in A \times B : x = a \}$

$= \{a\} \times B$

preimage of a

bad notation, if use (a,b) cannot make sense of $a=a$.

<u>Note</u>: π_1 is not injective <u>unless</u> B is a set of <u>one</u> element.

π_1^{-1} means <u>preimage</u> <u>NOT</u> inverse function