Last time: Vector space over $\mathbb{R}$. (real vector spaces)

Today: 1) Linear subspaces of a vector space.
2) Complex numbers.

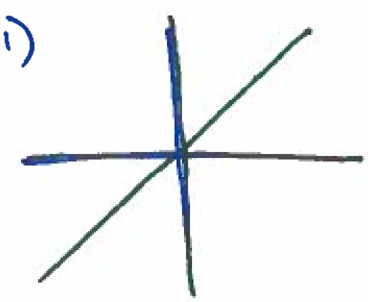
Let V be a vector space (over $\mathbb{R}$).

Def'n: A subset $W \subset V$ is called a linear subspace (vector subspace)

if: 1) $\forall\ x, y \in W \quad x+y \in W$
2) $\forall\ t \in \mathbb{R},\ x \in W \quad t \cdot x \in W$

"W is closed under addition and scalar multiplication"

Examples: 1)



$V = \mathbb{R}^2$
its linear subspaces are:
1) V itself
2) any line containing the origin $(0,0)$.
3) $\{(0,0)\}$ ← one point: the origin.

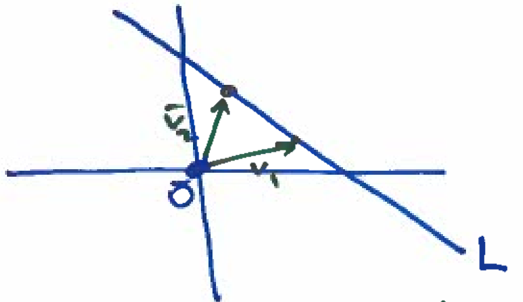
Remark: If $W \subset V$ is a linear subspace, then $W \ni \bar{0}$ ← $\boxed{\text{"W contains the element } \bar{0}\text{" same as } \bar{0} \in W}$

Pf: [Assume we know that $-x = (-1) \cdot x$ (Homework).]

Let $x \in W$. Then by (2), $(-1) \cdot x \in W$

By (1), $\underbrace{x + (-1) \cdot x}_{= \bar{0}} \in W$, so $\bar{0} \in W$.

Back to our example:



Let $W = L \cup \{(0,0)\}$

Why is this not a linear subspace?

$v_1, v_2 \in W$

$v_1 + v_2 \notin W.$

– picture proof of why W is not closed under addition.

<u>General proof of our example:</u>

Let $W \subset V$ be a linear subspace.

Suppose $W \neq \{\bar{0}\}$ ("non-trivial" subspace).

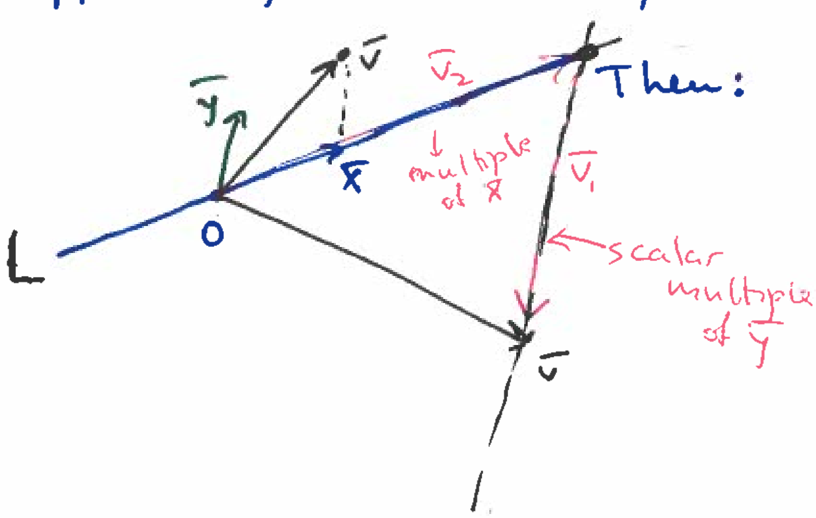
Let $\bar{x} \neq \bar{0}$ be an element of W.

Then $t \cdot \bar{x} \in W$ for all $t \in \mathbb{R}$

(so W has to contain the line through $\bar{0}$ and "tip of" $\bar{x}$)

if $W = \{t \cdot \bar{x} \mid t \in \mathbb{R}\}$, then W is a line through $\bar{0}$ and we are done.

Suppose $\bar{y} \in W$ and $\bar{y} \neq t \cdot \bar{x}$



Then: for any vector $\bar{v}$ in $\mathbb{R}^2$, there exist a, b s.t.

$$\bar{v} = a\bar{x} + b\bar{y}$$

↑ line parallel to $\bar{y}$ through ~~the~~ the tip of $\bar{v}$.

Find the ~~xxxxxx~~ intersection with L

This gives us $\bar{v}_1$ parallel to $\bar{y}$, $\bar{v}_2$ parallel to $\bar{x}$

such that $\bar{v} = \bar{v}_1 + \bar{v}_2$

we have: $\bar{v}_1 \in W$, $v_2 \in W$ by (2)

Then $\bar{v} \in W$. by (1).

we proved: if W contains $\bar{y} \neq t \cdot \bar{x}$ then $W = \mathbb{R}^2$.

Exer: Think of how to phrase this proof in terms of coordinates

(recall, formally $\mathbb{R}^2$ is the set of pairs (α, β) $\alpha, \beta \in \mathbb{R}$.

then given $\bar{x} = (x_1, x_2)$

$\bar{y} = (y_1, y_2)$

$\bar{v} = (\alpha, \beta)$

want to find $t_1, t_2 \in \mathbb{R}$ s.t. $\bar{v} = t_1 \bar{x} + t_2 \bar{y}$.

You will get a system of linear equations.

[Pay attention to the condition you need for this system to have exactly one solution !

(the condition on $\bar{x}, \bar{y}$].]

Complex numbers

Consider $x^2+1=0$, has no real solutions.

Recall: 0) at some point you knew only positive numbers (integers!)

-1 was such a number that $(-1)+1=0$.

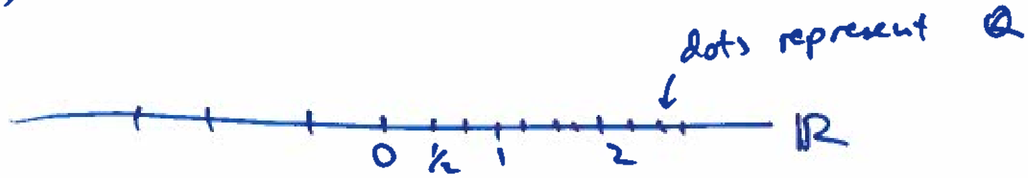
Cool thing: $\{0\}\cup\mathbb{N}\cup\{-1,-2,-3,\dots\}=\mathbb{Z}$

is closed under addition! and multiplicati

1) introduce: $\frac{1}{2}, \frac{1}{3}, \dots$

Get $\mathbb{Q}$ - rationals: closed under addition, multiplication and division (by $x\neq 0$).

2) $x^2-2=0 \rightsquigarrow \sqrt{2}$



"completion" of $\mathbb{Q}$ is $\mathbb{R}$ (Math 321).

↑ decimal expansions $0.11231129 7\dots$

3) Introduce $\mathbb{C}$: $\boxed{\text{declare } i=\sqrt{-1}}$ ← want it to be true

$$\mathbb{C}=\{a+bi \mid a,b\in\mathbb{R}\}.$$

Def: addition: $(a_1+b_1 i)+(a_2+b_2 i)=(a_1+a_2)+(b_1+b_2)i$

multiplication $(a_1+b_1 i)(a_2+b_2 i)$

$$=a_1a_2+b_1a_2 i+a_1b_2 i+\underbrace{b_1b_2 i^2}_{=-b_1b_2}$$

def of multiplication $\longrightarrow$ $=(a_1a_2-b_1b_2)+(b_1a_2+a_1b_2)i$

$\mathbb{C}$ is the set $\{a+bi \mid a, b \in \mathbb{R}\}$ with these rules for addition and multiplication.

Alternative way to think of it:

$\mathbb{C}$ is a "2-dimensional" vector space over $\mathbb{R}$ (i.e., $\mathbb{R}^2$) with ~~a~~ an additional operation:

~~[illegible]~~ $(a_1, b_1) \cdot (a_2, b_2) \stackrel{\text{def}}{=} (a_1a_2 - b_1b_2,\ b_1a_2 + a_1b_2)$

Question: Can you put "multiplication" structure on $\mathbb{R}^3$?

$\mathbb{R}^4$?