Last time: General fields.

- $\mathbb{R}$ - familiar
- $\mathbb{Q}$ - rationals, $\mathbb{Q} \subset \mathbb{R}$
- $\mathbb{C} \supset \mathbb{R}$, so we have $\underbrace{\mathbb{Q} \overset{\text{huge}}{\subset} \mathbb{R}} \overset{\text{"degree 2"}}{\subset} \mathbb{C}$

  new.
- $\mathbb{F}_p$ - finite fields of p elements
  $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$.

Remark: $\mathbb{F}_p \not\subset \mathbb{Q}$.
not a subfield!!

you can use integers to represent its elements but the operations are different.

Characteristic: weird thing happens:

$$\underbrace{1+1+1+1\ldots+1}_{p \text{ times}} = p \text{ in } \mathbb{Q}$$
$$= 0 \text{ in } \mathbb{F}_p$$

characteristic 0

$\mathbb{F}_p$ has "characteristic p"

The characteristic is either 0 ← you never get 0 by adding 1's.

or a prime number. (exer. or see the textbook).

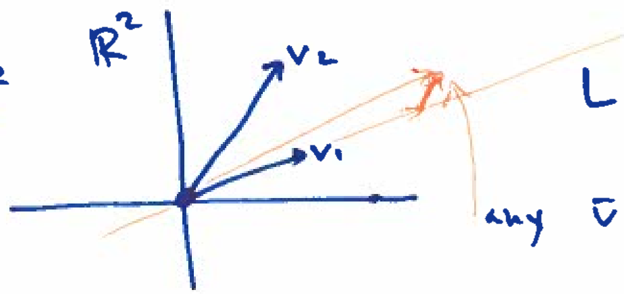
Today: Dimension of a vector space.

Let V be a vector space over a field $\mathbb{F}$.
(i.e., we can add vectors in V, and multiply by scalars from F).

Def. Let $v_1, \dots, v_n$ be elements of V.
Then $L(v_1, \dots, v_n) = \{ \lambda_1 v_1 + \dots + \lambda_n v_n \mid \lambda_i \in F \}$.

– the set of all linear combinations of $v_1, \dots, v_n$

Example



$L(v_1, v_2) = \mathbb{R}^2$

any $\bar{v}$ can be written as $\lambda_1 \bar{v}_1 + \lambda_2 \bar{v}_2$ $\lambda_{1,2} \in \mathbb{R}$
(2 lectures ago).

- Let v_1, v_2 be nonzero vectors in $\mathbb{R}^2$
$L(v_1, v_2) \neq \mathbb{R}^2 \iff v_1 \parallel v_2$ (parallel).
(we proved this already!)

Proposition For any $v_1, \dots, v_n \in V$
$L(v_1, \dots, v_n)$ is a linear subspace of V.

Pf: Need to check that $L(v_1, \dots, v_n)$ is closed under addition and scalar multiplication

- Let $x = \lambda_1 v_1 + \dots + \lambda_n v_n \in L(v_1, \dots, v_n)$
$+ \; y = \mu_1 v_1 + \dots + \mu_n v_n \in L(v_1, \dots, v_n)$

$x + y = (\lambda_1 + \mu_1) v_1 + \dots + (\lambda_n + \mu_n) v_n \in L(v_1, \dots, v_n)$

· similarly, for scalar mult.

<u>Convention</u>: $L(\emptyset) = \{\bar{0}\}$ ← single point subspace.

no v's.

"linear hull
linear span"

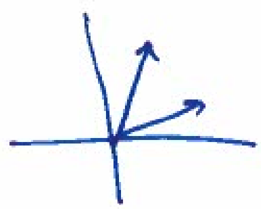
<u>Def</u>: Let $v_1, \dots, v_n \in V$.
The set $\{v_1, \dots, v_n\}$ is called <u>linearly independent</u>

if ($\lambda_1 v_1 + \dots + \lambda_n v_n = \bar{0} \iff \lambda_1 = \lambda_2 = \dots = \lambda_n =$

"no <u>non-trivial</u> linear combination of $\vec{v}_1, \dots v_n$ is zero")

<u>Example</u> in our $\mathbb{R}^2$ example:
$v_1, v_2 \neq 0$
v_1, v_2 are linearly independent $\iff$ if are <u>not</u> parallel

"if and only if"

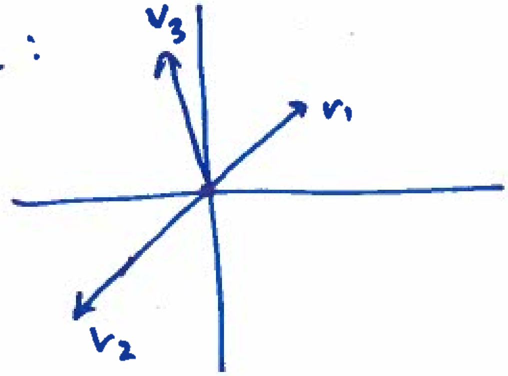


<u>Def</u>: $v_1, \dots, v_n$ are called <u>linearly dependent</u> if
there exist $\lambda_1, \dots \lambda_n \in F$, not all zero, such that
$\lambda_1 v_1 + \dots + \lambda_n v_n = \bar{0}$.
(ie., <u>not</u> linearly independent).

<u>Def</u>: (equivalent) you can express one of the v_i's as a linear combination of the others.

of "linearly dependent".

Example:



$\{v_1, v_2, v_3\}$ – linearly dependent:

we have

$$v_2 - \lambda_0 v_1 = 0$$

$$v_2 - \lambda_0 v_1 + 0 \cdot v_3 = 0$$

$$v_2 = \lambda_0 v_1$$

λ_0 – some negative number.

Lemma: If $\{v_1, \dots, v_k\}$ is a linearly dependent set of vectors in V

then any set of vectors that contains it is also linearly dependent.

Remark: 1) There are spanning sets – a set $v_1 \dots v_n$ s.t. $L(v_1, \dots, v_n) = V$

they do not have to be lin. indep.

If you take any spanning set, add some vectors into it, it will continue being a spanning set, but it will for sure be linearly dependent.

2) There are linearly independent sets.

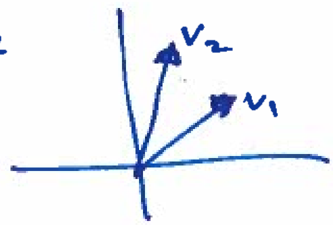
Any subset of a linearly independent set is linearly independent.

Independent sets do not have to span all of V,

but they always span some linear subspace of V.

Def (!) A basis for V is a set of vectors in V that is both spanning and linearly independent.

Example



$\leftarrow$ basis for $\mathbb{R}^2$.

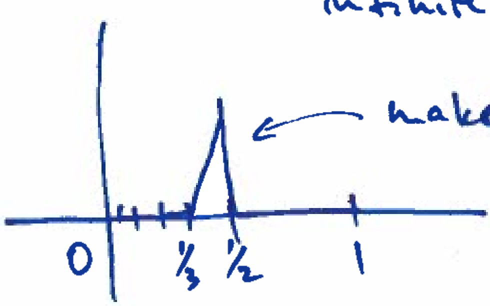
Main Theorem If V has a finite basis, then every basis of V has the same number of elements.

Def: • This number is called the dimension of V.

• If V does not have a finite basis, then V is called infinite-dimensional.

Examples: ① $V = \{f : \mathbb{R} \to \mathbb{R}\}$ – all functions
(or – all $\cup$ continuous fns
– all $\cup$ differentiable ___)

Why: to prove V is infinite-dimensional, all we have to do is make an infinite linearly indep. set in V



$\leftarrow$ make a function $\neq 0$ on $\left[\frac{1}{n+1}, \frac{1}{n}\right]$
$= 0$ elsewhere.

Exer: any finite collection of them is lin. indep.

② • $\mathbb{R}$ is infinite-dimensional as $\mathbb{Q}$-vector space

③ • $\mathbb{C}$ is 2-dim. over $\mathbb{R}$.