Math 223, Basics Quiz.

NAME

Short answer section: In Questions 1-3 no proof is required but if you want you can add a line of explanation.

1. Decide if the following statements are True/False.

   (a) Any vector space can be made into a field.

   False. There is really no reason why this should be true. If you want a counterexample with proof, however, this takes us outside the scope of this course; but for example, $\mathbb{R}^3$ cannot be made into a field : in fact, $\mathbb{C}$ is the only field containing $\mathbb{R}$ that is finite-dimensional as an $\mathbb{R}$-vector space (if you are curious why, ask me after class, the answer is long but within your reach).

   (b) If F is a field, it can be thought of as a 1-dimensional vector space over itself.

   True. By definition.

   (c) Any vector space over $\mathbb{C}$ can be thought of as a vector space over $\mathbb{R}$.

   True. This was a homework problem.

   (d) Any vector space over $\mathbb{R}$ can be thought of as a vector space over $\mathbb{C}$.

   False. For example, $\mathbb{R}$ itself is not a vector space over $\mathbb{C}$. We know that as in the question above, that any $\mathbb{C}$-vector space of dimension n is a $2n$-dimensional vector space over $\mathbb{R}$, so any real vector space of odd dimension *cannot* be a $\mathbb{C}$-vector space.

   (e) If U_1, U_2 are linear subspaces of a vector space V, then $U_1 \cap U_2$ is also a linear subspace.

   True. It is clearly closed under addition and multiplication by scalars.

   (f) If U_1, U_2 are linear subspaces of a vector space V, then $U_1 \cup U_2$ is also a linear subspace.

   False. In fact, we had a homework problem saying that it is a linear subspace if and only if one of U_1, U_2 is a subspace of the other.

   (g) $\mathbb{R}$ (with its usual operations of addition and multiplication) is an infinite-dimensional vector space over $\mathbb{Q}$.

   True. By the same *restriction of scalars* procedure that allowed us to think of any $\mathbb{C}$-vector space as an $\mathbb{R}$-vector space, we can think of $\mathbb{R}$ as a $\mathbb{Q}$-vector space (simply forget that you can multiply any two real numbers, and only remember that you can multiply any real number by any rational number; this allows you to think of the real numbers as elements of a vector space over $\mathbb{Q}$). The hard part was the matter of dimension. I stated that this space is infinite-dimensional in lecture, but without proof. Here are some ideas for the proof (though they take us outside of this course, into the domain of number theory).

For real numbers as vectors in a $\mathbb{Q}$-vector space, linear independence means the following: $x_1, \ldots, x_n \in \mathbb{R}$ are linearly independent if $c_1 x_1 + \ldots c_n x_n = 0$ for some $c_i \in \mathbb{Q}$ implies that all the scalars c_i are zero. How to find infinitely many linearly independent real numbers? One idea: take $\sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots$ – the square roots of prime numbers. It sounds plausible that they would be linearly independent over $\mathbb{Q}$ (as defined above). How to prove it formally: you can first try to prove that 1, $\sqrt{2}$ and $\sqrt{3}$ are linearly independent.

Another idea: for that you need to believe that π is *transcendental*, meaning that there is no polynomial equation with rational coefficients that π is a solution of: you cannot have $c_0 + c_1\pi + \ldots c_n \pi^n = 0$ with non-zero $c_0, \ldots, c_n \in \mathbb{Q}$. This means that the collection of real numbers $\{1, \pi, \pi^2, \ldots, \pi^n, \ldots\}$ of powers of π is linearly independent over $\mathbb{Q}$.

(h) $\mathbb{Q}$ (with its usual operations of addition and multiplication) is an infinite-dimensional vector space over the finite field $\mathbb{F}_p$.

False. Recall that in $\mathbb{F}_p$ the sum of p 1's is zero: $1 + 1 + \ldots 1 = 0$. Then, if we somehow made $\mathbb{Q}$ into an $\mathbb{F}_p$-vector space, we would have $px = 0$ for every $x \in \mathbb{Q}$, but of course this is impossible. (Recall: $\mathbb{F}_p$ is a field of *characteristic p.*)

(i) In any vector space, vectors v_1, v_2 and $v_3 = v_1 - v_2$ are linearly dependent.

True. v_3 is a linear combination of v_1 and v_2 by definition.

(j) In any vector space, any collection of vectors that contains $\overline{0}$ is linearly dependent.

True. We can take $\lambda \cdot \overline{0} = \overline{0}$ for some $\lambda \neq 0$, giving us a non-trivial linear combination that equals zero (the other vectors do not have to be involved in it).

(k) Any vector space over $\mathbb{R}$ contains at least three linearly independent vectors.

False. This says, *dimension of any real vector space is at least 3*, which is, of course, not true.

2. Find $(3 + 2i)^{-1}$ in $\mathbb{C}$.

Solution. Using the 'conjugate' trick:

$$(3 + 2i)^{-1} = \frac{3 - 2i}{3^2 + 4^2} = \frac{3}{13} - \frac{2}{13} i.$$

3. Describe all the linear subspaces of $\mathbb{R}^3$.

Solution. $\{\bar{0}\}$, all the lines containing the origin, all the planes containing the origin, and $\mathbb{R}^3$ itself.

Proof Question: proof required.

4. Let V be an n-dimensional vector space over a field F. Prove that a collection of n vectors $\{v_1, \ldots, v_n\}$ forms a basis of V if and only if it is linearly independent.

   Solution. The 'if' part: suppose $v_1, \ldots, v_n$ are linearly independent. We need to prove they form a basis. The only thing to prove is that they span V. Consider the space $L(v_1, \ldots, v_n)$ - the subspace of V spanned by the v_i. We need to prove that it equals V. If there were a vector $w \in V$ that does not lie in $L(v_1, \ldots, v_n)$, then, by basis extension theorem, we would have a basis of V that contains $v_1, \ldots, v_n$, and w. This means we would have a basis with at least $n+1$ elements, which is impossible since we are given that V is n-dimensional.

   The 'only if' part. We need to prove that if $\{v_1, \ldots v_n\}$ is a basis, then it is linearly independent. But it is part of the definition of a basis, so there is nothing to prove.