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Review

Linear maps

- $A: V \longrightarrow W \overset{\text{choose basis in } V, W}{\rightsquigarrow}$ matrix

  n-dim   m-dim

$$\begin{array}{c c} m & \begin{pmatrix} & & & \\ & & & \end{pmatrix} \\ & n \end{array} \longrightarrow \text{lin. map}, \; A: F^n \longrightarrow F^m$$

Kernel of lin. map $\longleftrightarrow$ null space of a matrix

$\longleftrightarrow$ the space of solutions to a system of eqns with $Ax = \bar{0}$

Image $\longleftrightarrow$ the space spanned by the col. of A

Tools:

Row reduction, det(A)

e.g.

Suppose

$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ $v_3 = \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix}$

For what $a \in \mathbb{R}$ do these v NOT form a basis of $\mathbb{R}^3$?

$\hookrightarrow$ check for linear dependence

- put in rows of a matrix $\rightarrow$ row reduction
  col.s

Find a basis for the space spanned by them?

- when put into col.s, we get a map: $\mathbb{R}^3 \longrightarrow V$ (space spanned by v_1 v_2 v_3)

$$\begin{cases} e_1 \longmapsto v_1 \\ e_2 \longmapsto v_2 \\ e_3 \longmapsto v_3 \end{cases}$$

Find a

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 0 & a \end{pmatrix} \underset{R_3 - R_1}{\overset{R_2 - 2R_1}{\rightsquigarrow}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & -1 & a-1 \end{pmatrix} \underset{R_2\, R_3}{\overset{\text{swap}}{\rightsquigarrow}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1-a \\ 0 & -2 & -1 \end{pmatrix} \overset{R_3 + 2R_2}{\rightsquigarrow} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1-a \\ 0 & 0 & 1-2a \end{pmatrix}$$

$1 - 2a = 0 \longrightarrow a = \frac{1}{2}$

alternatively, compute $\det(A) = 0$, expand in last row

$$\det A = 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + a \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 1 - 2a = 0 \longrightarrow a = \frac{1}{2}$$

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$\longrightarrow$ e.g.

Variation

$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ $\quad v_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

assume v_1, v_2, v_3 are lin. dep

consider the plane given by $\lambda v_1 + \mu v_2 + \nu v_3$

Find a basis for this plane $\longrightarrow v_1, v_2$

an equation for this plane (cross product)

$s v_1 + t v_2 \longrightarrow$ in $Ax + By + Cz + D = 0$

New question

$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad v_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

Consider 3-space

$r v_1 + s v_2 + t v_3 \longrightarrow a x_1 + b x_2 + c x_3 + d x_4 = 0$

- a point in space looks like

$$\begin{pmatrix} r+2s+2t \\ t \\ r \\ s+t \end{pmatrix}$$

$\longrightarrow a(r+2s+2t) + bt + cr + d(s+t) = 0 \quad \forall r, s, t$

$(a+c)r + (2a+d)s + (2a+b+d)t = 0$

$$\begin{cases} a+c = 0 \\ 2a+d = 0 \\ 2a+b+d = 0 \end{cases}$$

- consider matrix : $\mathbb{R}^3 \xrightarrow{A} \mathbb{R}^4$ (not surjective), $\operatorname{Im} A$ = parametric space

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$ * transpose

condition:

$b = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \operatorname{Im} A$

$\therefore \left(\begin{array}{ccc|c} 1 & 2 & 2 & x_1 \\ 0 & 0 & 1 & x_2 \\ 1 & 0 & 0 & x_3 \\ 0 & 1 & 1 & x_4 \end{array}\right) \xrightarrow{RR} \cdots$ equation = row of 0's

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Question:

Suppose

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ -1 & 2 & 7 \end{pmatrix}$$

find a basis for Ker A

- consider row reducing (augmented matrix)

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 2 & 7 & 0 \end{array}\right) \longrightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 5/2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\therefore \begin{cases} x_1 = 2t \\ x_2 = -\frac{5}{2}t \\ x_3 = t \end{cases} \longrightarrow t\begin{pmatrix} 2 \\ -\frac{5}{2} \\ 1 \end{pmatrix} \longrightarrow t\begin{pmatrix} 4 \\ -5 \\ 2 \end{pmatrix}$$