Self-adjoint linear operators

Def. Linear operator on a Euclidean space V (back to real vector spaces) $A : V \to V$ that satisfies:

$$(A(v), w) = (v, A(w)) \quad \forall\ v, w \in V$$

inner product in V

"self-adjointness" condition. is called self-adjoint.

- Facts: 1) self-adj lin. operators are given by symmetric matrices w.r.t. an orthonormal basis. (see below)

  2) Any self-adjoint lin. op. has an orthonormal basis of eigenvectors (so it is diagonalizable) - this is a version of the Spectral Theorem that we will prove.

Lemma: If v, w are eigenvectors of a self-adj. lin. op. $A : V \to V$ corresponding to distinct eigenvalues, then they are orthogonal.

Pf:
$$Av = \lambda v \qquad \lambda \neq \mu.$$
$$Aw = \mu w$$

compute $\langle Av, w\rangle = \langle \lambda v, w\rangle$

Self-adj $\longrightarrow$ $\|$

$\langle v, Aw\rangle = \langle v, \mu w\rangle$

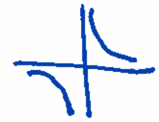
So $\langle \lambda v, w\rangle = \langle v, \mu w\rangle$

$$\lambda\langle v, w\rangle \qquad \mu\langle v, w\rangle$$

So if $\lambda \neq \mu$, then $\langle v, w\rangle = 0$. ∎

(So if A has distinct eigenvalues, we are done with proof of (2))

Theorem "Principal axes transformation" ← a historic name, had to do with the axes of a hyperbola.



- A self-adj. lin. op. in a Euclidean space has an orthonormal basis of eigenvectors

Today: Prove it; talk about consequences and a recipe for finding 'this' basis.

We already proved that if $A: V \to V$ is self-adjoint (ie. $(Av, w) = (v, Aw)$) then eigenvectors corresponding to distinct eigenvalues are orthogonal.

Corollary: If A has n distinct eigenvalues ($n = \dim V$) then A has a basis of eigenvectors that is orthonormal

(already orthogonal, normalize the eigenvectors to have length $= 1$).

Two issues:
1) Multiple eigenvalues: $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ ← cannot happen!
2) Existence of real eigenvalues.
(here we are in a real Euclidean space! what if all eigenvalues are in $\mathbb{C}$ not in $\mathbb{R}$?

doesn't happen for self-adjoint lin. op. in V.

Our proof will go by induction on $\dim(V)$.

Base case: $V \cong \mathbb{R}$ $\qquad A: \mathbb{R} \to \mathbb{R}$ (self-adjoint.)

Then $Ax = \lambda x$
$\uparrow$ eigenvalue (real by def.)

Nothing to prove.

normalize v to have length 1.

To prove the induction step, recall

<u>orthogonal complement</u> : if $v \in V$,

$$v^{\perp} = L(v)^{\perp} = \{ w \in V : w \perp v \}.$$

i.e. $(w, v) = 0$

linear subspace of V. (dim $= n-1$

(Recall: $v^{\perp} \oplus L(v) = V$,

We need

<u>Lemma</u> $A : V \to V$ - a self-adj. lin. op.

Suppose v is an eigenvector for A.

Then $v^{\perp}$ is an <u>invariant subspace</u> under A.

(More generally, if U is invariant for A, then $U^{\perp}$ is also invariant)

<u>Recall</u>: <u>Invariant subspace</u>: a subsp. $U \subset V$ s.t. $A(U) \subset U$.

(Ex A 1-dim. subspace of V is <u>invariant</u> under A $\iff$ it is spanned by an eigenvector of A).

<u>Proof of the Lemma</u> Need to prove: if $u \in v^{\perp}$ then $Au \in v^{\perp}$.

Meaning, want to check: $(v, Au) = 0$.

But $(v, Au) \underset{\text{self-adj}}{=} (Av, u) \underset{v \text{ is eigenvector}}{=} (\lambda v, u) =$

$= \lambda (v, u) \underset{u \in v^{\perp}}{=} 0$

So once we have an eigenvector v,

we have $V = L(v) \oplus \underbrace{v^{\perp}}_{= U} \leftarrow (n-1)$-dim space.

Because U is invariant, we can think of

$A|_{U}$ - "A restricted to U" - the lin. op. $A : U \to U$

So by induction assumption, $A|_U$ has an orthonormal basis of eigenvectors in U

The basis of V is $\left(\frac{v}{\|v\|}, \text{ the basis of } U \right)$

This completes the proof, if we knew that at least one eigenvector exists.

Why is this a question? - how do we know that there is a real eigenvalue?

We need one more

Lemma $A : V \to V$ a self-adj. lin. op. ← in fact, all eigenvals have to be real

Then it has a real eigenvalue

$\gamma, w \in \mathbb{R}$

Proof: We have a complex eigenvalue, $\lambda = \gamma + iw$

A corresponding eigenvector is $x + iy$ (imagine that V becomes $V_{\mathbb{C}}$ - a complex vector space)

vector of real parts

$A : V \to V$ ← real space

λ - a root of $\det(A - \lambda I)$ - a poly with real coeffs.

$(\lambda \in \mathbb{C})$

Action of A on $x+iy$:

$$A(x+iy) = (\gamma + iw)(x+iy)$$

means:

$$(*)\begin{cases} Ax = \gamma x - wy & \leftarrow \text{real part} \\ Ay = \gamma y + wx & \leftarrow \text{imaginary part} \end{cases}$$

$v_1 \ldots v_n$ - basis of V

look at

$V_{\mathbb{C}} = \{ z_1 v_1 + \ldots + z_n v_n \mid z_i \in \mathbb{C} \}$.

So: $$A(x+iy) = \overbrace{(\gamma + iw)}^{\lambda}(x+iy)$$

Still have $\langle Ax, y \rangle = \langle x, Ay \rangle$ ($x, y \in V$ - real vectors)

Plug in $(*)$

Get: $$\langle \gamma x - wy, y \rangle = \langle x, \gamma y + wx \rangle$$

$$\cancel{\gamma \langle x, y \rangle} - w \langle y, y \rangle = \cancel{\gamma \langle x, y \rangle} + w \langle x, x \rangle$$

$$w \cdot \underbrace{(\|x\|^2 + \|y\|^2)}_{\neq} = 0 \implies \boxed{w = 0}$$

so λ was real!

Consequences

Spectral Theorem If $A: V \to V$ is a self-adj. lin. op.

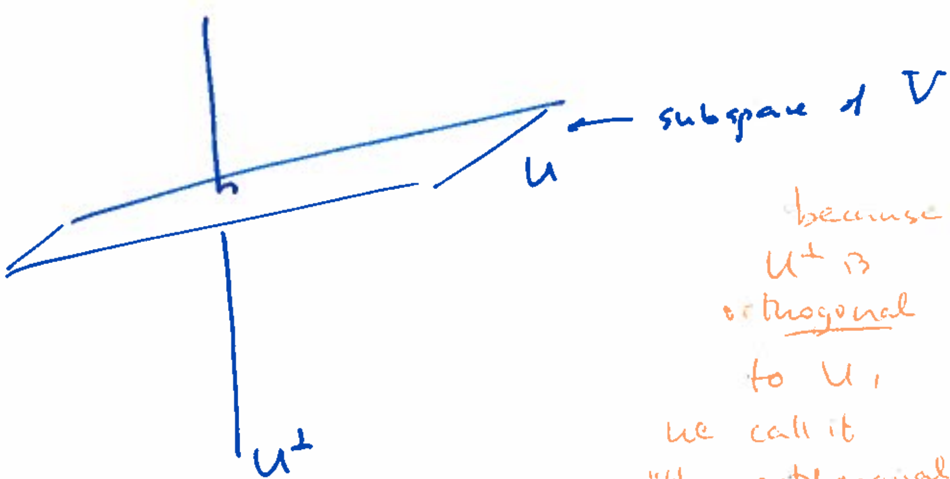
then we can write A as

$$A = \sum_{k=1}^{m} \lambda_k P_k$$

where $\lambda_1, \ldots, \lambda_m$ are eigenvalues of A

and $P_1, \ldots, P_m$ are the orthogonal projectors onto the corresponding eigenspaces.

Explanation of the statement



Recall: if we have

$$V = U \oplus U^\perp$$

can define the projector onto U along $U^\perp$.

because $U^\perp$ is orthogonal to U, we call it "the orthogonal projector" onto U.

$$P_U : V \to U$$

$$\operatorname{Ker}(P_U) = U^\perp$$

$$P_U^2 = P_U$$

Spectral Thm says: $E_i = \operatorname{Ker}(A - \lambda_i I)$ where λ_i are the eigenvalues of A

then $V = E_1 \oplus \cdots \oplus E_m$. P_i - projector onto E_i

$$A = \sum_{i=1}^{m} \lambda_i P_i \iff A = \begin{bmatrix} \boxed{\begin{matrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_1 \end{matrix}} & & \\ & \boxed{\begin{matrix} \lambda_2 & & \\ & \ddots & \\ & & \lambda_2 \end{matrix}} & \\ & & \boxed{\lambda_m} \end{bmatrix}$$

Important point What about the transition matrix? Suppose $\{e_i\}$ is an orthonormal basis in V.

What about the transition matrix that makes A diagonal?

! This matrix will be **orthogonal** (i.e. will preserve the inner product $\Leftrightarrow$ that it takes an orthonormal basis to an orthonormal basis)

This is because we have an **orthonormal** basis in which A is diagonal.

Important point: Self-adjoint, what does it mean in terms of matrices?

Let $\{e_i\}$ be an **orthonormal** basis of V.

$A : V \to V$ is self-adjoint $\Leftrightarrow$ its matrix with resp. to $\{e_i\}$ is **symmetric**.

think + read

Proof: We have a_{ij} = the ith entry in the j-th column

$$\begin{array}{cc} i & \begin{pmatrix} & \vdots & \\ \cdots & a_{ij} & \\ & \vdots & \end{pmatrix} \\ & j^{th}\text{ column} \end{array}$$

= the coefficient of $A(e_j)$ with respect to e_i

Since the basis $\{e_i\}$ is orthonormal, this coefficient is $(A(e_j), e_i)$

Then we get: $a_{ij} = (A(e_j), e_i) \underset{\text{self-adjoint}}{=} (e_j, A(e_i))$

$\underset{\text{same argument as above}}{=} a_{ji}$