The University of British Columbia

Math 223 — Honours Linear Algebra

2023, April 25

Surname: ____________________ First Name: ____________________ Student ID: __________

Instructions

• **Explain** your reasoning thoroughly, and **justify** all answers (even if the question does not specifically say so).
• NO calculators or other aids are permitted.
• Duration: **2.5** hours.

Good luck, and enjoy the break.

Question	Points	Score
1	27	
2	8	
3	5	
4	11	
5	14	
6	12	
7	8	
8	15	
Total:	100	

I. **Short Answer Problems.** Put your answer in the box provided **but show your work also**. Not all questions are of equal difficulty. Full marks will be given for correct answers placed in the box; partial credit might be given for incorrect answers.

1. Determine whether the statement is True or False. No proofs required, but anything illustrating your thinking might earn partial credit if the answer is incorrect. **If you recognize any named theorem, include its name.**

2 marks (a) Elementary row operations can be expressed using matrix multiplication.

Answer:

2 marks (b) If $A : \mathbb{R}^3 \to \mathbb{R}^3$ is given by the matrix $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ in the standard basis, then there does not exist a basis of $\mathbb{R}^3$ in which the matrix of A is diagonal.

Answer:

2 marks (c) If $A : \mathbb{R}^3 \to \mathbb{R}^3$ has characteristic polynomial $p_A(t) = (t-1)(t-2)(t-3)$, then there exists a basis of $\mathbb{R}^3$ in the matrix of A is diagonal.

Answer:

3 marks (d) Given a linear operator $A : V \to W$, we can choose a basis in V and a basis in W such that the matrix of A with respect to these bases is diagonal with only 1s and 0s on the diagonal, with the number of 1s equal to the rank of A.

Answer:

3 marks (e) Given a linear operator $A : V \to V$, there exists a linear transformation $C : V \to V$ such that $C^{-1}AC$ is diagonal.

Answer:

3 marks (f) Let $A : \mathbb{R}^n \to \mathbb{R}^n$. If A is injective, then A is surjective.

Answer:

3 marks

(g) Let $A : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathrm{rk}(A) = n$. Then $\det(A) \neq 0$.

Answer:

3 marks

(h) For any $A : \mathbb{R}^n \to \mathbb{R}^n$ there exists $B : \mathbb{R}^n \to \mathbb{R}^n$, such that AB is surjective.

Answer:

3 marks

(i) Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. If $A^2 = 0$ then $A = 0$.

Answer:

3 marks

(j) If V is a space of dimension n over the field $\mathbb{F}_p$ of p elements, and W is a space of dimension m over $\mathbb{F}_p$, then $V \times W$ is a linear space consisting of $p^n p^m$ elements.

Answer:

2. Let F be a field. Decide if U is a linear subspace of a vector space V in all the questions below. (You can assume that V is a linear space). If U is a linear subspace, determine its dimension.

$\boxed{\text{2 marks}}$ (a) $V = M_{n\times n}(F)$ – the space of all $n \times n$-matrices with entries in F;

$$U = \{A \in V \mid A^{-1}\text{exists}\}.$$

$\boxed{\text{2 marks}}$ (b) $V = M_{n\times n}(F)$,

$$U = \{A \in V \mid A^t A = \mathrm{Id}\}.$$

$\boxed{\text{4 marks}}$ (c) $V = M_{2\times 2}(F)$, and let $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

$$U = \{A \in V \mid AB = BA\} - \text{the set of matrices commuting with } B.$$

3. Let

$$A = \begin{bmatrix} 1 & 2+i \\ -1 & 3 \end{bmatrix}$$

3 marks (a) Find A^{-1}. Write the answer in the form that does not use i in any denominators.

2 marks (b) Is A diagonalizable?

II. Long Answer questions. In the following questions, include complete proofs and show all your work.

$\boxed{\text{6 marks}}$ 4. (a) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Find a basis for $\ker(A)$ and for $\mathrm{Im}(A)$.

5 marks

(b) Let V be the $\mathbb{R}$-vector space of all polynomials with real coefficients. Find the dimension of the subspace of V spanned by the polynomials

$$\begin{aligned} p_1 &= (1+x) + 2x^2 + 3x^3, \\ p_2 &= 4(1+x) + 5x^2 + 6x^3, \\ p_3 &= 7(1+x) + 8x^2 + 9x^3. \end{aligned}$$

Include a complete proof!

5. Let A be the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$\boxed{\text{3 marks}}$ (a) Find the characteristic polynomial of A.

$\boxed{\text{3 marks}}$ (b) You can assume that the roots of the characteristic polynomial of A are $\lambda_1 = 5$ and $\lambda_2 = -1$ with multiplicity 2. Find a basis for each of the eigenspaces of A.

4 marks (c) Find an *orthogonal* matrix P such that $P^{-1}AP$ is diagonal.

4 marks (d) Find A^{25} (you can leave the answer as a product of matrices).

6. Let U be the plane in $\mathbb{R}^3$ (with the standard inner product) defined by the equation

$$x + y + z = 0$$

2 marks (a) Find a basis for U.

4 marks (b) Find an orthonormal basis for U.

2 marks (c) Find an orthonormal basis of $\mathbb{R}^3$ that contains a basis of U.

4 marks (d) Let $P_U : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projector onto U. Write a matrix for P_U in the standard basis of $\mathbb{R}^3$ (you can leave it as a product of matrices).

This page is left blank for your work

8 marks 7. Let V be a Euclidean space, and let W be a subset of V. Prove that W contains a basis of V if and only if the following condition holds:

If $v \in V$ satisfies $\langle v, w \rangle = 0$ for all $w \in W$, then $v = 0$.

8. In this problem, please do not use methods from multivariable calculus. Consider the space $\mathbb{R}^3$ with the standard inner product.

$\boxed{\text{4 marks}}$ (a) Find the maximum of the function $f(x,y,z) = x+y+z$ on the unit sphere (the unit sphere is defined by the equation $x^2+y^2+z^2=1$).

$\boxed{\text{3 marks}}$ (b) Find a linear transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$ that takes the unit sphere to the ellipsoid defined by $x^2+4y^2+9z^2=1$ (in the standard coordinates).

$\boxed{\text{3 marks}}$ (c) Define an inner product on $\mathbb{R}^3$ such that the unit sphere with respect to the new inner product is defined by the equation $x^2+4y^2+9z^2=1$ in the standard coordinates.

$\boxed{\text{5 marks}}$ (d) Find the maximum of the function $f(x,y,z) = x+y+z$ on the ellipsoid defined by $x^2+4y^2+9z^2=1$.