Math 223, Homework 2: Vector spaces. Due Monday September 25.

1. Let V be a real vector space. Prove that:
   (a) Prove that for $x \in V$, $0 \cdot x = \overline{0}$.
   (b) Prove that $\lambda x = \overline{0}$ if and only if $x = \overline{0}$ or $\lambda = 0$.
   (c) Prove that $(-1)x = -x$.
2. Problem 2.2 from Jänisch
3. Problem 2.3 from Jänisch
4. (a) Prove that all the functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the differential equation
   $$f''(x) - f(x) = 0$$
   form a vector space.
   (b) Prove that the solutions to this differential equation satisfying $f(\pi) = 0$ form its linear subspace.
   (c) Now consider the differential equation $f'(x) = f(x)^2$. Do its solutions form a linear subspace of the space of all differentiable functions?
5. (a) Let S be the set of all functions $f : [0, 1] \to \mathbb{R}$ such that f is integrable and $\int_0^1 f(x)\, dx = 0$. Is the set S a linear subspace of the space of all functions?
   (b) Let W be the set of integrable functions from $[0, 1] \to \mathbb{R}$ defined by the condition $\int_0^1 f(x)\, dx = 1$. Is the set W a linear subspace of the space of all functions?
6. Find all the complex numbers z such that $z^3 = 1$.
7. Prove that the set $F := \{a + b\sqrt{5} | a, b \in \mathbb{Q}\} \subset \mathbb{R}$ forms a field (with respect to the operations of addition and multiplication that it inherits from $\mathbb{R}$) (it is denoted by $\mathbb{Q}(\sqrt{5})$).