Math 223, Homework 3: Linear dependence; bases.

1. Let V be a vector space over a field F. Prove that the a set of vectors $\{v_1, \ldots, v_n\}$ forms a basis of V if and only if the following condition holds:

   for every $v \in V$, there exists *unique* collection of coefficients $\lambda_1, \ldots, \lambda_n \in F$ such that
   $$v = \lambda_1 v_1 + \cdots + \lambda_n v_n.$$

2. Problem 3.1 from Jänisch

3. Problem 3.2 from Jänisch

4. If U_1 and U_2 are complementary subspaces of V (see Jänisch, Problem 3.2), then we write $V = U_1 \oplus U_2$. Prove that $V = U_1 \oplus U_2$ if and only if the following condition holds:

   for every $v \in V$, there exist *unique* vectors $v_1 \in U_1$ and $v_2 \in U_2$ such that $v = v_1 + v_2$.

5. Let V be an n-dimensional vector space over the field of complex numbers $\mathbb{C}$. Consider it as a real vector space (with the same operation of addition, and for scalar multiplication, only pay attention to the multiplication by the real scalars; this is called *restricting the scalars*). What is the dimension of V as a vector space over $\mathbb{R}$?

   (*Hint:* First, think of a 1-dimensional vector space over $\mathbb{C}$; then use the last problem of the supplementary problems below; or use Problem 1 above).

Supplementary problems, do not hand in.

1. From Curtis: Problems 4.1, 4.3 and 4.4 on pp. 32-33.

2. Using the notation of the Problem 4, prove that $\{v_1, \ldots, v_n\}$ is a basis of V if and only if
   $$V = L(v_1) \oplus \cdots \oplus L(v_n).$$