Homework 4: Linear transformations, Part 1.

1. Let V and W be vector spaces over a field F. Let $A : V \to W$ be a linear transformation that has an inverse function $B : W \to V$. Prove that B has to be a linear transformation.

2. Problem 4.1 from Jänisch

3. Think of $\mathbb{C}$ as a 2-dimensional vector space V over $\mathbb{R}$, and let $A : V \to V$ be the linear transformation of V given by the multiplication by $1 + 2i$ in $\mathbb{C}$. Write the matrix of A with respect to the standard basis of V.

4. Let $V = \mathbb{R}^n$. Prove that every linear functional $f : V \to \mathbb{R}$ is of the form $f(x_1, \ldots, x_n) = a_1x_1 + \cdots + a_nx_n$ for some constants $a_1, \ldots, a_n \in \mathbb{R}$.

   (Hint: think of what it does to the standard basis vectors).

   Remark: This shows that, in fact, the space $V^* := \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ of all linear functionals on V is isomorphic to V (for every finite-dimensional space V). For infinite-dimensional spaces this is, generally, not true.

5. Consider the linear space V polynomials of degree not greater than n over a field $F \subset \mathbb{C}$ (that is, the space of all functions $f : F \to F$ of the form $f(x) = a_nx^n + \cdots + a_1x + a_0$, where $a_0, \ldots, a_n \in F$). Let $D : V \to V$ be the linear map $D(f) = f'$ (the derivative). Find the kernel and image of D.

6. Let V be an arbitrary vector space over a field F, and let $P : V \to V$ be a linear operator with the property that $P^2 = P$ (here by P^2 we mean P composed with itself). Such linear operators are called *projectors.*

   (a) Prove that $V = \ker(P) \oplus \mathrm{Im}(P)$.

   (b) Make an example of such a linear operator on $\mathbb{R}^3$.