Sep 18, 2023

· Last time: defined vector spaces, linear subspaces

Today: · some easy consequences of the axioms.

· complex numbers.

Next time: · general fields, odds and ends on vector spaces.

Proving things from the definition of a vector space:

1) The axiom says, $\exists\, \bar{0} \in V: \quad x + \bar{0} = x \quad \forall x \in V.$

Can there be two zero elements?

No: (proof by contradiction):

suppose we had two zeroes: $\bar{0}_1$ and $\bar{0}_2$.

Then: $\bar{0}_1 = \bar{0}_1 + \bar{0}_2 = \bar{0}_2 + \bar{0}_1 = \bar{0}_2$

because $\bar{0}_2$ is a zero.

b/c $\bar{0}_1$ is a zero.

So $\bar{0}_1 = \bar{0}_2$.

Similarly: for every x there is only one $(-x)$.

We will use the notation $x-y$ for $x+(-y)$.

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2) Example: $t \in \mathbb{R}$, then $t \cdot \bar{0} = \bar{0}$

wrong way to prove it:

$\bar{0} = (0, \ldots, 0)$

$t \cdot \bar{0} = (t \cdot 0, \ldots, t \cdot 0)$

$= (0, \ldots, 0) = \bar{0}$

Works for $\mathbb{R}^n$ which is just one example of a vector space.

Need to do: proof from the axioms.

Pf: Let $t \in \mathbb{R}$. Want to prove: $t \cdot \bar{0} = \bar{0}$.

checking that $t \cdot \bar{0}$ satisfies def'n of "zero"

Means: need to prove that for $\forall x \in V$, $x + t \cdot \bar{0} = x$

Two cases: $t = 0$ or $t \neq 0$.

Consider $t \neq 0$ case first.

def of $\bar{0}$

$$x + t \cdot \bar{0} = \left(t \cdot \tfrac{1}{t}\right) \cdot \bar{x} + t \cdot \bar{0} = t \cdot \left(\tfrac{1}{t} x + \bar{0}\right)$$

$\mathbb{R}$

(using $1 \cdot \bar{x} = x$)

by axiom (7)

$$= t \cdot \tfrac{1}{t} x = x$$

So $t \cdot \bar{0}$ satisfies the def'n of "zero", then it equals $\bar{0}$

because we just proved that zero is <u>unique</u>.

3) <u>Case 2</u>: $t=0$.

Let's prove a more general statement then:

$0 \cdot x = 0$ for <u>all</u> $x \in V$, not just for $\bar{0} \in V$.

$\boxed{\text{homework}}$

4) Similarly (see homework also) we prove that the element $-x$ is unique for every $x \in V$, so we can define the difference

$x - y \in V$ for $x, y \in V$ as

$$x - y \overset{\text{def}}{=} x + (-y).$$

etc.

(see other easy properties of vector spaces in the book)

Complex numbers

Consider $x^2+1=0$, has no real solutions.

Recall: 0) at some point you knew only positive numbers (integers!)

-1 was such a number that $(-1)+1=0$.

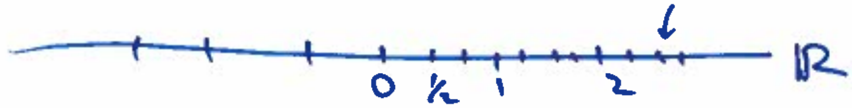
cool thing: $\{0\}\cup\mathbb{N} \cup \{-1,-2,-3\dots\} = \mathbb{Z}$
is closed under addition! and multiplication

1) introduce: $\frac{1}{2}, \frac{1}{3}, \dots$

Get $\mathbb{Q}$ - rationals: closed under addition, multiplication and division (by $x\neq 0$).

2) $x^2-2=0 \rightsquigarrow \sqrt{2}$

dots represent $\mathbb{Q}$



"completion" of $\mathbb{Q}$ is $\mathbb{R}$ (Math 321).

↑ decimal expansions $0.112311297\dots$

3) Introduce $\mathbb{C}$: $\boxed{\text{declare } i=\sqrt{-1}}$ ← want it to be true

$$\mathbb{C} = \{a+bi \mid a,b\in\mathbb{R}\}.$$

Def: addition: $(a_1+b_1 i)+(a_2+b_2 i) = (a_1+a_2)+(b_1+b_2)i$

multiplication $(a_1+b_1 i)(a_2+b_2 i)$

$$= a_1a_2 + b_1a_2 i + a_1b_2 i + \underbrace{b_1b_2 i^2}_{=-b_1b_2}$$

def of multiplication $\longrightarrow$ $= (a_1a_2 - b_1b_2) + (b_1a_2 + a_1b_2)i$

$\mathbb{C}$ is the set $\{a+bi \mid a, b \in \mathbb{R}\}$ with these rules for addition and multiplication.

<u>Alternative way to think of it</u>:

$\mathbb{C}$ is a "2-dimensional" vector space over $\mathbb{R}$ (i.e., $\mathbb{R}^2$) with an additional <u>operation</u>:

$$(a_1, b_1) \cdot (a_2, b_2) \overset{\text{def}}{=} (a_1 a_2 - b_1 b_2\,,\; b_1 a_2 + a_1 b_2)$$

<u>Question</u>: Can you put "multiplication" structure on $\mathbb{R}^3$?

$\mathbb{R}^4$?