# Vector spaces

**Example (Main)**

Think about $\mathbb{R}^n = \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_{n \text{ times}}$

$= \{(a_1, \dots, a_n) \mid a_i \in \mathbb{R}\}.$

We can do: addition of $n$-tuples

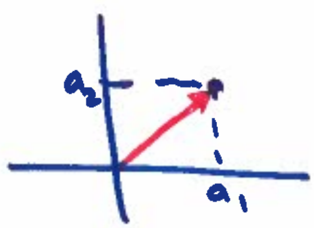
$$(a_1, \dots, a_n) + (b_1, \dots, b_n) = (a_1 + b_1, \dots, a_n + b_n)$$

and: scalar multiplication:

$$t \in \mathbb{R} \qquad t \cdot (a_1, \dots, a_n) = (ta_1, \dots, ta_n)$$

in pictures:

$\mathbb{R}^2$




can represent a tuple by a vector

(for now, a vector for us is just the tuple of its components.

$\bar{v} = \langle a_1, \dots, a_n \rangle$

We will use $(a_1, \dots, a_n)$ instead.

Some sources use $\hat{v}$, $\bar{v}$ to denote vectors.

We just say: let $x \in \mathbb{R}^n$ be a vector, $x = (x_1, \dots, x_n)$

**Note:** for now we do not think of a vector as "magnitude + direction" - do not yet have the notion of either. Later, we'll define extra structure on a vector space, which will give us magnitudes and angles ("Euclidean spaces").

Want to move from this example to an abstract notion of a vector space.

<u>Def</u>: A <u>vector space</u> is a set $V$ with two operation
over $\mathbb{R}$
"over the set of real numbers"
field

<u>addition</u>: for every $x, y \in V$, an element $x+y \in V$

<u>scalar multiplication</u>: for $t \in \mathbb{R}$, $v \in V$
$t \cdot v \in V$.

<u>satisfying the axioms</u>:

addition axioms

1) $(x+y)+z = x+(y+z)$ – associativity of addition
2) $x+y = y+x$ – commutativity of addition
3) There exists $\bar{0} \in V$ s.t. $\forall x \in V$ $x + \bar{0} = x$ (existence of 0)
4) $\forall x \in V$, $\exists (-x) \in V$ s.t. $x + (-x) = \bar{0}$
(existence of additive inverse)

5) $\forall t \in \mathbb{R}$, $s \in \mathbb{R}$, $x \in V$ $t \cdot (sx) = (ts) \cdot x$
6) $1 \cdot x = x$ for all $x \in V$
7) $\forall t \in \mathbb{R}$, $x, y \in V$ $t \cdot (x+y) = t \cdot x + t \cdot y$
8) $\forall t, s \in \mathbb{R}$, $x \in V$ $(t+s) \cdot x = t \cdot x + s \cdot x$

linearity

<u>Remark</u>: Our example $\mathbb{R}^n$ satisfies these axioms. ($\bar{0} = (0, \dots, 0)$).

1. Axioms say: "exists $\bar{0} \in V$". Could there be more than one such element?

Will discuss this on Wednesday.

**Def** Let $V$ be a vector space. A set $W \subset V$ is called a linear subspace of $V$ if it is closed under addition and mult. by scalars: $\forall\ x \in W,\ t \in \mathbb{R}$ $\quad t \cdot x \in W$

$\forall\ x, y \in W \quad x + y \in W$.

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**Example of a vector space** (Important!)

Let $V$ be the $\overset{\text{set}}{\text{space}}$ of all functions $f : \mathbb{R} \to \mathbb{R}$

Define addition: $(f + g)(x) = f(x) + g(x)$ – a function from $\mathbb{R} \to \mathbb{R}$

scalar multiplication: $\underbrace{(t \cdot f)}_{\text{new function} : \mathbb{R} \to \mathbb{R}}(x) = t \cdot f(x)$

Exer check that this satisfies the def. of a vector space.

(Note: the zero in this space is the function: the constant zero function:

$z(x) = 0$ for all $x \in \mathbb{R}$

graph of $z(x)$.

**Example**: Consider the space of all differentiable functions $f : \mathbb{R} \to \mathbb{R}$. It is a linear subspace of the space of all functions.

In it, the solutions of a linear differential equation will form a linear subspace

(see homework 2 for detail)

<u>Example</u>: Linear subspaces of $\mathbb{R}$:

Let $W$ be a linear subspace of $\mathbb{R}$.

<u>claim</u>: $W = \{0\}$ or $W = \mathbb{R}$.

<u>Proof</u> Suppose $W \subset \mathbb{R}$ is a linear subspace and $W \neq \{0\}$.

Then there exists an element $x_0 \in W$ s.t. $x_0 \neq 0$. We claim that then every real number $y$ is also in $W$.

Indeed, since $x_0 \neq 0$, we can write

$y = \left(\frac{y}{x_0}\right) \cdot x_0$. The number $t := \frac{y}{x_0} \in \mathbb{R}$

is a scalar. Then by definition of a linear subspace, $t \cdot x_0 \in W$, so

$\frac{y}{x_0} \cdot x_0 = y \in W$. QED.