Today: • Multiplicativity of determinants

• transpose of a matrix

• Cramer's rule.

Proposition: A, B - $n \times n$-matrices

$$\boxed{\text{Then } \det(AB) = \det(A)\det(B).}$$

Pf: fix B, think of $\det(AB)$ as a function of A.

(in the book) it satisfies axioms of det, except for: the value at Id is $\det(B)$.

Then it has to be $\det(A)\det(B)$.

Another proof (same idea, spelled differently)

imagine a "unit cube" in V

volume 1

$\rightsquigarrow$ apply B

becomes a parallelepiped of volume $\det(B)$

$\vec{\ }$ apply A

volume gets scaled by $\det(A)$.

volume gets scaled by $\det(AB)$

Transpose matrix

given A (an $m \times n$ -matrix), we can "transpose" it:

interchange rows $\leftrightarrow$ columns

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix} \longrightarrow A^t = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix}$$

3×4

"A transpose"

4×3 - matrix

1.

- Transposition does not change det!
  (for square matrices)
- so everything I said about ~~column~~ row ops applies to column ops as well,
  and when thinking of volumes, could instead make a box of column vectors.

Reminder: this is equivalent to solving n systems of linear equations:

$$Ax_i = e_i = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow i^{th} \text{ place} \quad \text{for } i=1,\dots,n$$

$x_i = A^{-1}e_i$ — becomes the i^{th} column of A^{-1}.

Today: Cramer's rule – formula for finding a solution for a system of linear equations $Ax = b$.

(A – a square matrix, assume A is invertible $\Leftrightarrow \det(A) \neq 0$ so we know the solution exists and is unique! $x = A^{-1}b$.)

We have:

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Imagine that $\begin{pmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{pmatrix}$ is the solution to this system of equations.

Take x_i^0 for one fixed value of i — the i^{th} component of the solution vector

Consider the matrix

$$M_i = \begin{pmatrix} a_{11} & \dots & (x_i^0 a_{1i} - b_1) & \dots & a_{1n} \\ \vdots & & & & \\ a_{n1} & \dots & (x_i^0 a_{ni} - b_n) & \dots & a_{nn} \end{pmatrix}$$

multiplied the i^{th} column of A by x_i^0 and subtract the RHS.

Claim: det of this matrix M_i is 0.
(because x_i^0 is the ith component of the solution.)

Get:

$$x_i^0 \det \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} - \det \begin{pmatrix} a_{11} & \cdots & \boxed{} & \cdots & a_{1n} \\ a_{n1} & \cdots & \boxed{} & \cdots & a_{nn} \end{pmatrix} = 0$$

ith column of A replaced by the vector $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

So

$$\boxed{x_i^0 = \frac{\det \begin{array}{c} \text{ith column is } b \\ \begin{pmatrix} a_{11} \cdots & \boxed{} & a_{1n} \\ a_{n1} \cdots & \boxed{} & a_{nn} \end{pmatrix} \end{array}}{\det(A)}}$$

Cramer's rule.

Pf of Claim: our system of equations says:

$$a_{11} x_1^0 + a_{12} x_2^0 + \boxed{\cdots} + a_{1n} x_n^0 = b_1$$
$$\vdots$$
$$a_{n1} x_1^0 + a_{n2} x_2^0 + \boxed{\cdots} + a_{nn} x_n^0 = b_n$$

x_i^0 appears in: $a_{1i} x_i^0$
$\vdots$
$a_{ni} x_i^0$

so we have: $M_i \begin{pmatrix} x_1^0 \\ \vdots \\ -1 \\ \vdots \\ x_n^0 \end{pmatrix} = 0$, i.e. $\ker(M_i) \neq \{0\}$.
so $\det(M_i) = 0$.

in the ith place

$$= \frac{(-1)^{i+j} \det A_{ij}}{\det(A)}$$

throw away j^{th} column and i^{th} row

expand the numerator in the i^{th} column

a formula for the j^{th} entry in the i^{th} column of A^{-1}.

$$(A^{-1})_{ji} =$$

this is why you need to transpose

$$\text{So } A^{-1} = \frac{1}{\det(A)} \left((-1)^{i+j} \det(A_{ij}) \right)^{t}.$$

Wednesday Nov. 1

Today: Inverting matrices

- 6.4? "the adjugate matrix" - a ~~det~~ formula for A^{-1} that uses

this is §7.2 → Cramer's rule - a formula for the solution to $Ax=b$ (system of linear equations) using det.

Recall: our algorithm for finding A^{-1} for a square matrix A.

- A^{-1} is a matrix such that $AA^{-1} = A^{-1}A = Id$

also the matrix of the inverse linear operator (so should exist if and only if $A: V \to W$ is an isomorphism. $\iff \begin{cases} \dim V = \dim W & (A \text{ is a square matrix}) \\ \ker(A) = \{0\} \end{cases}$

$\Updownarrow$ A is injective

$\iff \begin{cases} \dim(V) = \dim(W) \\ rk(A) = n \iff A \text{ is surjective} \end{cases}$

Let A be $n \times n$-matrix.
To find A^{-1} we did:

$$\left(A \,\middle|\, \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix}\right) \underset{\substack{\text{elem.}\\ \text{row}\\ \text{operations}}}{\rightsquigarrow} \left(\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \,\middle|\, A^{-1}\right)$$

augmented matrix

↑ Id

(No column operations)

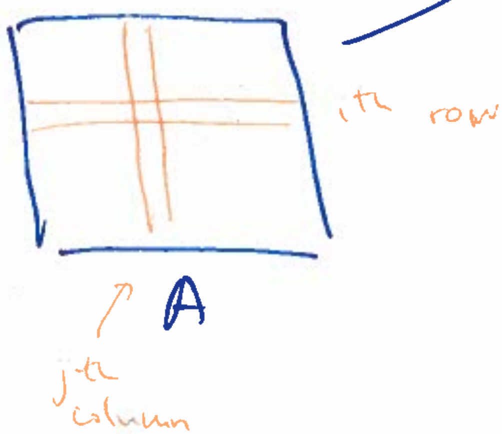
↑ if didn't get a pivot in each column, then A^{-1} doesn't exist.

A formula for A^{-1}

Assume A^{-1} exists $\Leftrightarrow \det(A) \neq 0$. ($A$ is an $n\times n$-matrix)

build the "adjugate" matrix of A.

Let $A_{ij} = \cancel{\det}\left(\square\right)_{(n-1)\times(n-1)}$



throw away the i^{th} row and the j^{th} column.

Get an $(n-1)\times(n-1)$ matrix.

Let $B = \left((-1)^{i+j} \det(A_{ij}) \right)^t$ ← each entry is the det of a "minor" of A (the matrix A_{ij}) (up to sign)

Then $B = A^{-1} \cdot \det(A)$.

$$\boxed{A^{-1} = \frac{\left((-1)^{i+j}\det(A_{ij})\right)^t}{\det(A)}}$$

Example (Memorize the result!)

for 2×2-matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad A_{ij} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

with signs: $\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

Now transpose: $\left((-1)^{i+j}A_{ij}\right)^t = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Then $\boxed{A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$

Check: $A^{-1}A = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$= \frac{1}{ad-bc}\begin{pmatrix} da-bc & db-bd \\ -ca+ac & -cb+ad \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\overset{\shortparallel}{0}$

Why does this formula work

The book: take $\left((-1)^{i+j}A_{ij}\right)^t \cdot A$

(in 6.4?)

see the diag. entries are $\det(A)$

by the formula for expansion of $\det(A)$

by the i^{th} ~~row~~. column

off-diagonal entries are 0 because get

matrices with 2 identical columns

(requires thought.)

Our proof: use Cramer's rule

the i^{th} column of A^{-1} is the vector of

solutions to $Ax = e_j = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ ← j^{th} place.

use Cramer's rule: $x_i = \dfrac{\det \begin{array}{c} \\ \\ \end{array}\begin{pmatrix} a_{11} & & \overset{i^{th}\text{ column}}{0} & & a_{1n} \\ & & 1 \leftarrow j^{th}\text{ place} & & \\ a_{n1} & & 0 & & a_{nn} \end{pmatrix}}{\det A}$

the i^{th} component of the solution.

Now use the cofactor expansion in the j^{th} column for the determinant in the numerator: all the terms equal 0 except for the one that corresponds to the "1", which is in row j and column i.

When we throw away the column and row of this 1, we get the minor A_{ji}.

We obtain: c_{ij} = the i^{th} component of the solution to $Ax = e_j$

the entry of A^{-1} in row i and column j

$$= \frac{(-1)^{i+j} \det(A_{ji})}{\det(A)}.$$

The formula for A^{-1} is just this statement written in the matrix form.

Answering a question: why do we care about this formula?

It is not convenient for computations. However, it is very important for understanding how A^{-1} changes if we perturb A slightly. Namely, you can expect that A^{-1} doesn't change much under such perturbation as long as $\det(A)$ is not close to 0.

If $\det(A)$ is close to 0, since it is in the denominator, the error of A^{-1} corresponding to a small change in A could be very large.