Today:

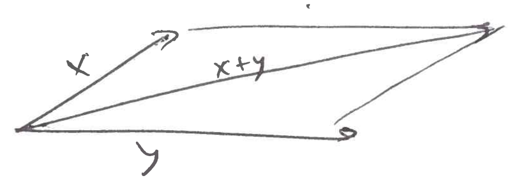
- Norms in vector spaces (Euclidean norms)
- Gram-Schmidt orthogonalization (orthonormalization).

What is a norm?

- Notion of length.

When V is just a vector space, a norm on V is a function $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ that satisfies:

1) $\|x\| = 0 \iff x = 0$
2) $\|\lambda x\| = |\lambda| \|x\|$
3) triangle: $\|x+y\| \leq \|x\| + \|y\|$



Recall we have been talking of Euclidean spaces: spaces with inner product:

$$(\, , \,) : V \times V \to \mathbb{R}$$

Given an inner product, we can define the norm:

$$\|x\| = \sqrt{(x,x)}$$

Such norms are called Euclidean norms

In the same way as we have defined other notions abstractly, we can just define ~~a~~ the notion of a norm axiomatically: it is any function from V to $\mathbb{R}_{\geq 0}$ satisfying:

Properties of norms

- Recall $\|x\| = \sqrt{(x,x)}$ - exists b/c $(x,x) \geq 0$ by def'n.

The norm satisfies the properties:

1) $\|x\| = 0 \iff x = 0$, otherwise $\|x\| > 0$.
2) $\|\lambda x\| = |\lambda| \, \|x\|$
3) $\|x+y\| \leq \|x\| + \|y\|$ - triangle inequality

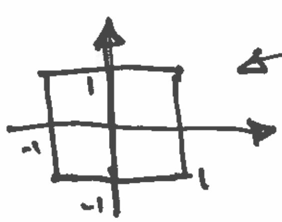
aside: any function from V to $\mathbb{R}$ that satisfies these properties is called a norm on V.

Our norms are Euclidean norms (came from an inner product).

There are many other norms, example on $\mathbb{R}^2$:

let $|x| = \max(|x_1|, |x_2|)$
(x_1, x_2)

what is the unit circle ? $= \{x : \|x\| = 1\}$



- a square!

Proof of triangle inequality for Euclidean norms :

Consider $\|x+y\|^2 = (x+y, x+y) = (x,x) + 2(x,y) + (y,y)$

Cauchy-Schwartz $\rightarrow$ $2(x,y) \leq 2\|x\|\,\|y\|$

$\leq \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$.

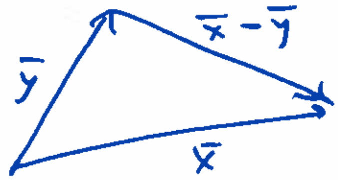
Does every norm come from an inner product?

No. Example: $\|x\| = \max(|x_1|, |x_2|)$ on $\mathbb{R}^2$ does not.

(unit sphere is not smooth).

For a Euclidean norm, (x, x) is a smooth function of the coordinates, so the unit sphere would be smooth.

Why is it called triangle inequality:



each side $\leq$ sum of lengths of the other two sides.

$$\|x\| \leq \underset{a}{\|y\|} + \underset{b}{\|x-y\|}$$

$$\|a+b\| \leq \|a\| + \|b\|.$$

also,

$\|x-y\| \leq \|x\| + \|y\|$

"triangle inequality" says: $\|x-y\| \leq \|x\| + \|-y\|$
$= \|x\| + \|y\|$

Now back to Euclidean spaces.

Orthogonal complements

Let V be a Euclidean space.

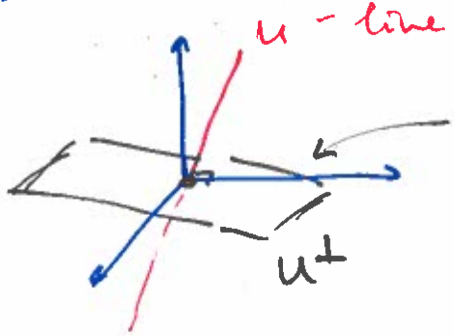
Let U be a subspace.

Def: $U^\perp$ "the orthogonal complement of U"

$= \{ v \in V : (v, u) = 0 \text{ for all } u \in U \}$

(the set of all vectors perpendicular to all vectors in U)

Example: $\mathbb{R}^3$, $(x,y) = x \cdot y$

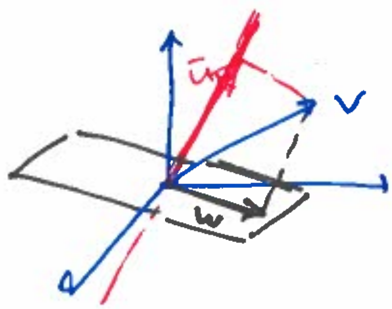


Theorem: Let U be a linear subspace of a Euclidean space V.

Then $V = U \oplus U^{\perp}$

(which means, each vector $v \in V$ has a unique decomposition $v = u + w$, where $u \in U$, $w \in U^{\perp}$)

Example: $\mathbb{R}^3$



We will prove this decomposition next time, along with the existence of orthonormal bases.

For this, we need to develop the Gram-Schmidt orthonormalization process.

Gram-Schmidt

1) <u>Recall</u>: <u>orthonormal</u> basis in a Eucl. sp. V:

$\{e_1, \dots, e_n\}$ - basis s.t.

$$(e_i, e_j) = \delta_{ij} \overset{def}{=} \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

"Kronecker δ"

Why do such bases exist?

Main point: In an orthonormal basis, the component of any vector v along e_i is (v, e_i)

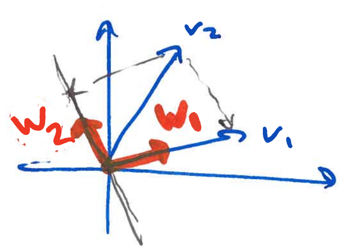
(<u>Recall</u>: $v = \sum_{k=1}^{n} \lambda_k e_k$

$$\underline{\underline{(v, e_i)}} = \left(\sum_{k=1}^{n} \lambda_k e_k, e_i\right) = \sum_{k=1}^{n} \lambda_k \underbrace{(e_k, e_i)}_{\substack{= \\ 1 \; k=i \\ 0 \text{ otherwi}}}$$

$$= \underline{\underline{\lambda_i}}$$

<u>Gram-Schmidt</u>: start with any basis V. Make it into an orthonormal basi

<u>Illustration on $\mathbb{R}^2$</u>:



<u>Given</u>: v_1, v_2 - lin. indep.

<u>Step 1</u>: scale v_1 to become a unit vector. $w_1 = \dfrac{v_1}{\|v_1\|}$

* <u>Step 2</u>: Replace v_2 with the orthogonal to w_1 component of v_2.

$$\tilde{v}_2 = \text{orth}_{v_1} v_2 = (v_2 - \text{proj}_{w_1} v_2)$$

Step 3: rescale $\tilde{v}_2$ to be a unit vector. Call it w_2

In Step 2, how to do it algebraically:

we know: component of v_2 along w_1

is (v_2, w_1)

Then $\text{proj}_{w_1} v_2 = \underset{\substack{\uparrow \\ \text{scalar}}}{(v_2, w_1)} \cdot \underset{\substack{\uparrow \\ \text{vector}}}{w_1}$ - vector parallel to w_1

How do we know that $v_2 - \text{proj}_{w_1} v_2$ is orthogonal to w_1?

Proof: $(v_2 - (v_2, w_1) w_1 , w_1)$

$$= (v_2, w_1) - (v_2, w_1) \underbrace{(w_1, w_1)}_{=1} = 0.$$

Now we do this in general.

To do this, we'll need to project vectors onto subspaces spanned by $w_1, \ldots, w_k$.

General strategy:

- Write a formula for the projector onto a space U along $U^\perp$ (the orthogonal projector) assuming U has an orthonormal basis
- Just as in $\mathbb{R}^2$, we will use this formula to replace the $(k+1)^{st}$ basis vector with a vector orthogonal to span$(w_1, \dots, w_k)$.

Will happen next class

off hr zoom 8-9