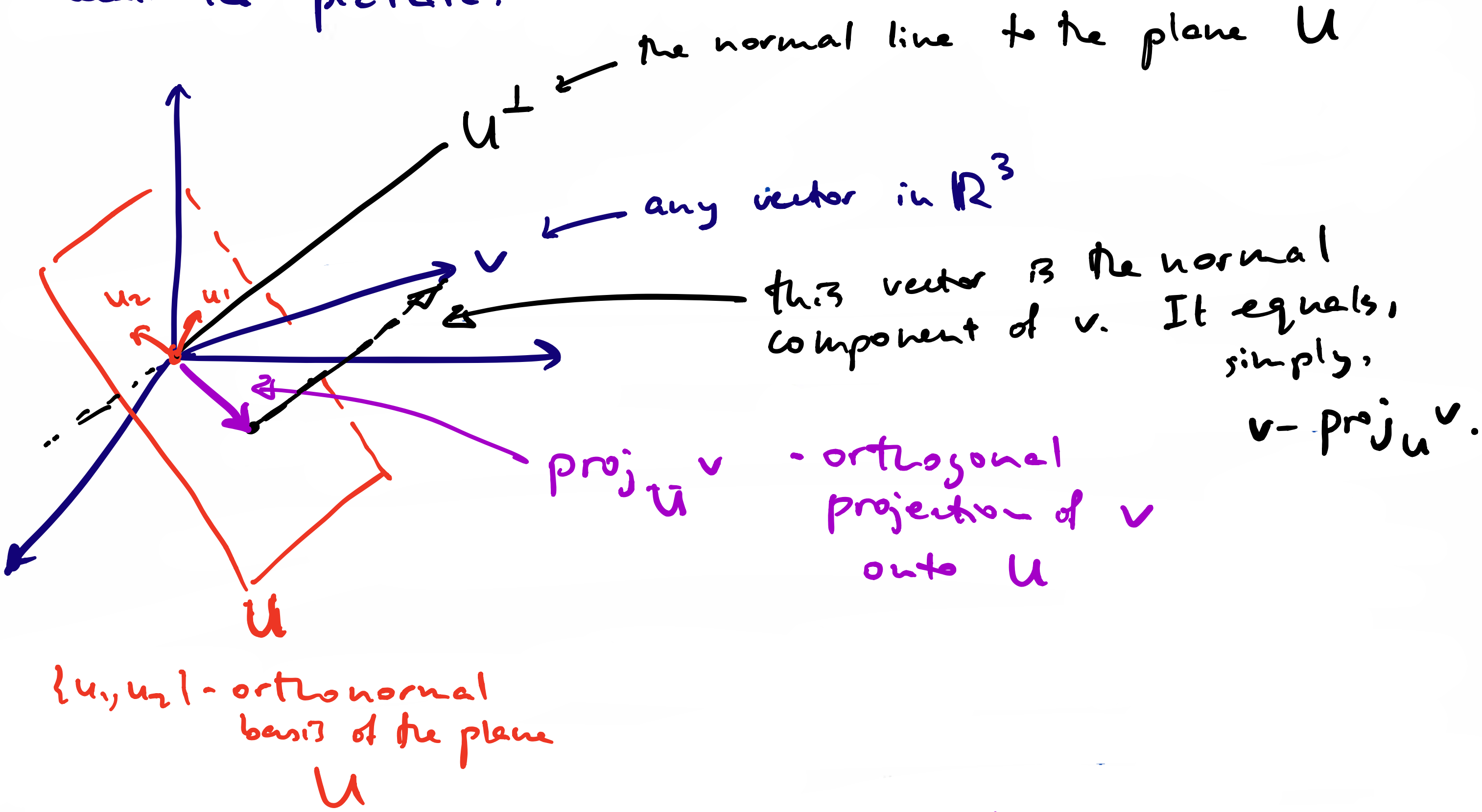
Today:
- orthogonal projections
- Gram-Schmidt orthogonalization
- row rank = column rank explained.

Recall the picture:



Here is how to express it algebraically:

Suppose we had an orthonormal basis $u_1, \ldots, u_k$ in U.

Then $\text{Proj}_U v = \sum_{i=1}^{k} (v, u_i) u_i$

orthogonal projection onto U

$$= \underbrace{(v, u_1)}_{\text{comp. of } v \text{ along } u_1} u_1 + \ldots + \underbrace{(v, u_k)}_{\text{comp. of } v \text{ along } u_k} u_k \in U$$

To prove: $v - \text{Proj}_U v$ is orthogonal to U.

We compute: $\left(v - \sum_{i=1}^{k} (v, u_i) u_i \,,\, \underset{\substack{\uparrow \\ U}}{u}\right) =$

$$= (v, u) - \sum_{i=1}^{k} (v, u_i)(u_i, u)$$

write $u = \sum_{i=1}^{k} a_i u_i$ - decomposition of u into the components along u_i in U.

$$= \sum_{i=1}^{k} a_i (v, u_i) - \sum_{i=1}^{k} (v, u_i)\underbrace{(u_i, u)}_{= a_i} = 0.$$

why $(u, u_i) = a_i$

Plug in $u = \sum_{i=1}^{k} a_i u_i$:

$$(u, u_j) = \left(\sum_{i=1}^{k} a_i u_i, u_j\right)$$

$$= \sum_{i=1}^{k} a_i \underbrace{(u_i, u_j)}_{= \delta_{ij}} = a_j \cdot 1$$

This says: all terms are 0 except for the j-th term

The j-th term is $a_j \cdot 1$

Gram - Schmidt orthogonalization process

Given a basis $\{v_1, \dots, v_n\}$ in V, how to make it into an orthonormal basis?

- Do it inductively:

1) • make v_1 a unit vector by replacing it with

$$\tilde{v}_1 = \frac{v_1}{\|v_1\|}.$$

2) Replace v_2 with $\tilde{v}_2' = v_2 - \mathrm{Proj}_{\tilde{v}_1} v_2 = v_2 - (v_2, \tilde{v}_1)\tilde{v}_1.$

$$= v_2 - \underbrace{(v_2, \tilde{v}_1)\tilde{v}_1}$$

works b/c $\tilde{v}_1$ is a unit vector

So: $\tilde{v}_2' \perp \tilde{v}_1$.

3) Make $\tilde{v}_2'$ into a unit vector:

$$\tilde{v}_2 = \frac{\tilde{v}_2'}{\|\tilde{v}_2'\|} = \frac{v_2 - (v_2, \tilde{v}_1)\tilde{v}_1}{\|v_2 - (v_2, \tilde{v}_1)\tilde{v}_1\|}$$

4) Let $\tilde{v}_3' = v_3 - \mathrm{Proj}_{L(\tilde{v}_1, \tilde{v}_2)} v_3$

$$= \underbrace{v_3 - \left((v_3, \tilde{v}_1)\cdot\tilde{v}_1 + (v_3, \tilde{v}_2)\cdot\tilde{v}_2\right)}_{\text{not } 0 \text{ b/c } v_1, v_2, v_3 \text{ are lin. indep.}}$$

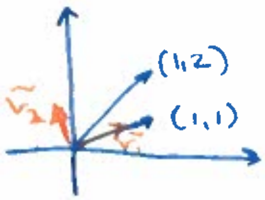
Let $\tilde{v}_3 = \dfrac{\tilde{v}_3'}{\|\tilde{v}_3'\|}$

...

This way we build an orthonormal basis, and it finishes the proof of the theorem that any Euclidean space has orthonormal bases.

See an example on the next page.

Example:



$v_1 = (1,1)$
$v_2 = (1,2)$

the usual dot product:

Process: 1) make v_1 unit: $\tilde{v}_1 = \frac{v_1}{\|v_1\|} = \boxed{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}$

2) $\tilde{v}_2' = v_2 - \mathrm{proj}_{\tilde{v}_1} v_2 = (1,2) - \tilde{v}_1 \cdot (v_2, \tilde{v}_1)$

right direction but not unit yet

$$= (1,2) - \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \cdot \frac{3}{\sqrt{2}} = (1,2) - \left(\frac{3}{2}, \frac{3}{2}\right)$$
$$= \left(-\frac{1}{2}, \frac{1}{2}\right)$$

3) make $\tilde{v}_2'$ unit:

$$\tilde{v}_2 = \frac{\tilde{v}_2'}{\|\tilde{v}_2'\|} = \boxed{\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}$$

- From this, we can prove that in a Euclidean space V, if U is a linear subspace, $V = U \oplus U^\perp$

orthogonal complement $= \{v \in V : (v,u) = 0 \ \forall u \in U\}$

Proof: We just showed that every $v \in V$ can be written as $v = \underbrace{\mathrm{proj}_U v}_{U} + \underbrace{(v - \mathrm{proj}_U v)}_{U^\perp}$.

Such decomposition is unique be $U \cap U^\perp = \{0\}$.

- Now we can go back to justify: row rank = column rank.

- A real finite-dim. vector space can be made into a Euclidean space (using dot product in $\mathbb{R}^n$).

The point: the space spanned by the columns of A = the image of the linear operator A.

$A: V \to V$
$\updownarrow$
matrix A w.r.t. our chosen basis.

Think of V as $\mathbb{R}^n$ with the standard basis.

With respect to the dot product, then the space spanned by the rows of A is $\ker(A)^\perp$

Important note: $\mathrm{Im}(A)$, $\ker(A)$ do not depend on the choice of basis in V.

The matrix for A does.

The notion of "$\perp$" also depends on the choice of (,)

So: this works when the choice of basis agrees with (,) (namely, is orthonormal).

(e.g.: standard basis + the usual dot product works).

Why does this work:

$$\begin{array}{c} r_1 \\ \\ r_n \end{array} \begin{pmatrix} a_{11} & - & - & a_{1n} \\ & & & \\ a_{n1} & - & - & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \underset{\text{def of } A\cdot x}{=} \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ - \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n \end{pmatrix}$$

row vectors

$$= \begin{pmatrix} r_1 \cdot x \\ \vdots \\ r_n \cdot x \end{pmatrix}$$

using the dot product

(this means, we are using the "standard basis" to write the matrix)

Now: $x \in \ker A \iff Ax = 0$

$$\iff \begin{cases} r_1 \cdot x = 0 \\ \vdots \\ r_n \cdot x = 0 \end{cases} \iff x \in (\text{row space } = L(r_1 \dots r_n))^\perp$$

So we proved: $\boxed{\ker A = L(r_1 \dots r_n)^\perp}$

Exer: $(U^\perp)^\perp = U$.

Then $(\ker A)^\perp = (L(r_1, \dots, r_n)^\perp)^\perp = L(r_1, \dots, r_n)$.

Then we have: row space $= \ker(A)^\perp$

column space $= \mathrm{Im}(A)$.

$V = \ker(A) \oplus \ker(A)^\perp$ ← our theorem applied to $U = \ker(A)$

Then $\boxed{\dim((\ker A)^\perp) = \dim(V) - \underbrace{\dim(\ker(A))}_{\text{"nullity" of } A}}$

but we also have $\mathrm{rk}(A) = \dim(\text{column space})$
$= \dim(\mathrm{Im}(A))$

and: dimension formula

$\dim V = \dim(\mathrm{Im}\, A) + \dim(\ker(A))$

so $\boxed{\text{column rank} = \dim V - \dim(\ker(A))}$

More about matrices and Euclidean spaces:

Orthogonal matrices and isometries (§8.3).

Def: Let V, W be Euclidean spaces (with inner products $(\,,\,)_V$ and $(\,,\,)_W$ respectively.)

An isometry between V and W is a linear operator $A: V \to W$ s.t.

$$(Av_1, Av_2)_W = (v_1, v_2)_V$$

for all $v_1, v_2 \in V$.

"preserves ~~metric~~" distances

More about isometries next time. Please read 8.3!

Exer: A is an isometry $\iff$ $\forall v \in V$
$\|v\|_V = \|Av\|_W$

hint: think of $(v+w, v+w) = \|v+w\|^2$

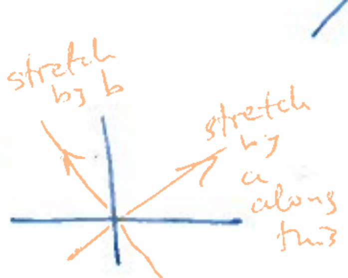
Def: A square matrix A is called orthogonal if $AA^t = Id$

Proposition: let V be a Euclidean space
let $\{v_1, \dots, v_n\}$ be an orthonormal basis in V
Then $A : V \to V$ is an isometry $\iff$
The matrix of A with respect to this basis
is orthogonal.

Pf: exer. (see the book)

Example: isometries in $\mathbb{R}^2$:
usual dot product

recall: bijective the linear transf : $\mathbb{R}^2 \to \mathbb{R}^2$

- Id – an isometry.
- rotations  – isometries
- reflections  – isometries
- 

dilations – not isometries
- shear transf. ← see the book (or not) – not isometries