Today: 1) isometries (orthogonal transformations) (8.3)

2) Eigenvalues and eigenvectors (9.1-9.2)

Recall: last time we defined isometries:
\[ A : V \to W \text{ is an isometry if } \forall x, y \in V, \quad (Ax, Ay) = (x, y) \]

Today: 1) some properties of isometries.

1) "Isometry" means: preserves length

Prop: \( A \) is an isometry \( \iff \) \( \forall v \in V \)
\[ \| Av \|_W = \| v \|_V \]

Proof: think of \( (v+w, v+w) = \| v+w \|^2 \)
we can write it as
\[ \| v+w \|^2 = (v+w, v+w) = (v, v) + 2(v, w) + (w, w) \]
\[ = \| v \|^2 + 2(v, w) + \| w \|^2 \]

Is \( A \) preserves lengths, then \( \| A(v+w) \|^2 \)
\[ = \| v+w \|^2 \]
\[ = \| v \|^2 + 2(Av, Aw) + \| w \|^2 \]
and we get that \( 2(Av, Aw) = 2(v, w) \).

Picture: \( v \rightarrow w \) if \( \| v \| \) and \( \| w \| \) are preserved, the length of the third side is preserved, the angle \( \alpha \) must be preserved, too.
**Definition:** A square matrix $A$ is called **orthogonal** if $AA^t = I_d$

**Proposition:** Let $V$ be a Euclidean space, let $(v_1, \ldots, v_n)$ be an orthonormal basis in $V$. Then $A : V \to V$ is an isometry. ($\square$)

The matrix of $A$ with respect to this basis is orthogonal.

**Proof:** It is part of the theorem below the example.

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**Example:** Isometries in $\mathbb{R}^2$:

- **Reflections**
  - $R_{\text{horizontal}} : \mathbb{R}^2 \to \mathbb{R}^2$
  - $R_{\text{vertical}} : \mathbb{R}^2 \to \mathbb{R}^2$

- **Rotations**
  - $\text{Id} : \mathbb{R}^2 \to \mathbb{R}^2$
  - Reflections

- **Dilations**
  - $\text{Id} \times k : \mathbb{R}^2 \to \mathbb{R}^2$
  - $\text{Id} \times k : \mathbb{R}^2 \to \mathbb{R}^2$

- **Shear Transform**
  - See the book (or not)

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**Note:** The above examples illustrate various types of isometries in two-dimensional space. Isometries preserve distances and angles, and they are fundamental in geometry and linear algebra.
We will be able to classify all \( \text{lin. transf.} \colon \mathbb{R}^2 \to \mathbb{R}^2 \)
(and even generally \( F^n \to F^n \)) without complete proof.

Prop: \( A \colon V \to V \)

\[ A \text{ is orthogonal (i.e., an isometry)} \implies A \text{ is bijective} \]

\[ \ker(A) = \{0\} \]

and \( \text{rk}(A) = \text{dim} V \)

Pf: \( (Av, Av) = (v, v) \)

So \( Av = 0 \implies (Av, Av) = 0 \implies (v, v) = 0 \implies v = 0. \)

So \( \ker(A) = \{0\} \). Then \( \text{rk}(A) = \text{dim} V \).

Theorem: The following conditions are equivalent:

1. \( A \colon V \to V \) is an isometry
2. \( A \) takes an orthonormal basis to an orthonormal basis

Now let \( A \) also be the matrix of this linear transform with respect to an orthonormal basis.

4. \( AA^t = \text{Id} \)
5. \( A^t A = \text{Id} \)
6. The columns of \( A \) form an orthonormal basis
7. The rows of \( A \) form an orthonormal basis

Sketch of the proof: \( (1) \implies (2) \) - by definition

\( (2) \implies (1) \): Suppose \( \{e_1, \ldots, e_n\} \) is an orthonormal basis.
we want to prove: \( (v, w) = (Av, Aw) \) for all \( v, w \in V \).

Let \( v = \sum_{i=1}^{n} a_i e_i \) and \( w = \sum_{i=1}^{n} b_i e_i \). (because the basis \( \{e_i\} \) is orthonormal)

Then \( (v, w) = \sum_{i=1}^{n} a_i b_i \).

Thus \( (Av, Aw) = \left( \sum_{i=1}^{n} a_i A(e_i), \sum_{i=1}^{n} b_i A(e_i) \right) \)

\[= \sum_{i,j=1}^{n} a_i b_j (A(e_i), A(e_j)) = \sum_{i=1}^{n} a_i b_i = (v, w) \]

by (2)

(2) \( = \) (6) - by definition of a matrix of a linear transform. (the columns of \( A \) are the images of the basis vectors)

(6) \( \iff \) (5) - columns of \( A \) are rows of \( A^t \).

Thus, \( A^t A = \left[ \begin{array}{c} \text{ith column of } A \\ \text{jth column of } A \end{array} \right] \left[ \begin{array}{c} A_t \\ J_A \\ \vdots \text{jth col. of } A \end{array} \right] \)

\[= \left[ \delta_{ij} \right] = I_d \]

because columns of \( A \) are orthonormal and of length 1.
(5) (c) (4) by taking transposes
(7) (a) (6): $A$ is orthogonal (5) $A^t$ is orthogonal (by (4) and (5))
so we can apply our statement to $A^t$,
and get that $A^t$ has orthogonal columns
$\therefore A$ is an isometry.
But the columns of $A^t$ are the rows of $A$.

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Corollary (1) If $A$ is an orthogonal matrix,
then $\det(A) = \pm 1$.

Note: If $\det(A) = 1$, $A$ is called special orthogonal.

Notation
- $\mathbf{On}(\mathbb{R}) = \{ A \in M_n(\mathbb{R}) : A \text{ is orthogonal} \}$
- the group of orthogonal matrices
- $\mathbf{SO}_n(\mathbb{R})$ - the group of special orthogonal matrices.
(see § 8.4. - optional reading)

(2) $A$ is orthogonal $\iff A^{-1} = A^t$
(This is because $AA^t = \text{Id}$)

Note: This means an interesting thing about minors of an orthogonal matrix:
remember the formula for $A^{-1}$ using the adjugate
Next topic: eigenvalues, eigenvectors

"Spectral Theory" - the study of eigenvalues/eigenvectors.

Applications: everywhere!

Differential equations, quantum mechanics, signal processing, Fourier transform, computer science, AI, …
Suppose $A: V \rightarrow V$ - given lin. op. (from $V$ to itself)

**Def.** A vector $v \in V$ is called an **eigenvector** for $A$ with **eigenvalue** $\lambda \in F$ if $Av = \lambda v$.

We will prove next class that they exist if $F = \mathbb{C}$. Over arbitrary fields, a given lin. op. might not have eigenvalues in that field.

**Our goal:** Figure out how to find them when they exist.
And use them.

**Example:** for $A = \lambda \cdot \text{Id}$, every vector $v \in V$ is an eigenvector with eigenvalue $\lambda$.

**Main observation:** if $v \in V$ is an eigenvector for $A$ with eigenvalue $\lambda$, then $v \in \ker (A - \lambda \text{Id})$

indeed, if $Av = \lambda v$, then $Av - \lambda v = 0$,

$$ (A - \lambda \text{Id})v $$

Then $\det (A - \lambda \text{Id}) = 0$

I write the matrix for $A - \lambda \text{Id}$ in any basis
(det does not depend on the choice of basis.)

Then the way to find eigenvalues is solve the equation $\det (A - \lambda \text{Id}) = 0$ for $\lambda$.

**Example** Let $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$ - mean, the lin. op. given by this matrix in the standard basis of $\mathbb{C}^2$.

Let's find its eigenvalues and eigenvectors.
The matrix of $A - \lambda I$ (w.r.t. the standard basis)

$$
\begin{bmatrix}
1 - \lambda & 4 \\
1 & 1 - \lambda
\end{bmatrix}
$$

$$
\det \begin{bmatrix}
1 - \lambda & 4 \\
1 & 1 - \lambda
\end{bmatrix} = (1 - \lambda)^2 - 4 = 1 - 2\lambda + \lambda^2 - 4
= \lambda^2 - 2\lambda - 3.
$$

We want: the values of $\lambda$ for which it is 0!

$$
\lambda_{1,2} = \frac{2 \pm \sqrt{4 + 12}}{2} = \frac{2 \pm 4}{2} = -1 \text{ or } 3.
$$

(This is why we are working over $C$; over $\mathbb{R}$, we don't have to have roots!)

This one has real roots.

How to find eigenvectors:

Now that we know the eigenvalues, we can make a system of equations:

for $\lambda_1$: $Av = \lambda_1 v$, or better: $(A - \lambda I)v = 0$.

\[-1: \begin{bmatrix}
1 - (-1) & 4 \\
1 & 1 - (-1)
\end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

Solve as usual: \[
\begin{bmatrix}
2 & 4 \\
1 & 2
\end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

It should have infinitely many solutions! (If it doesn't, your "eigenvalue" is wrong).

$x_1 + 2x_2 = 0$, so $x_1 = -2x_2$.\]
Pick a vector that spans $\text{ker} \ (A - \lambda I_d)$.
Here, $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ works.

(Note: if $v$ is an eigenvector for $A$ with eigenvalue $\lambda$, then $a \cdot v$ with $a \neq 0$ also satisfies this.)

For $\lambda_2$: \[
\begin{bmatrix}
1 & -3 \\
1 & 1-3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

$x_1 - 2x_2 = 0 \quad x_1 = 2x_2$

get: $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = v_2$.

So: for our $A$, $\lambda_1 = -1$, eigenvector is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$; $\lambda_2 = 3$, eigenvector is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

The matrix for $A$ in the basis $\{v_1, v_2\}$ is $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ - diagonal with eigenvalues on the diagonal.

So our $A$ has an eigenbasis.

Next time: in general.
Recall: \( A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \). We figured out:
\[ \lambda_1 = -1 \quad \text{eigenvalue} \]
\[ \lambda_2 = 3 \quad \text{eigenvalue} \]
with eigenvectors \( v_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \)
determined \( v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \)
up to a scalar.

What are eigenvectors geometrically?

We had \( A : \mathbb{R}^2 \to \mathbb{R}^2 \)

in the \( (v_1, v_2) \) basis, \( A \) has matrix:
\[ \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \]
eigenvalues on the diagonal.

How do we know \( A \)? — We could try to say what it does.

or we give it by a matrix —
the standard basis

(this is what we did).

Does \( A \) stabilize any line?

The line doesn’t move.

Any such line is a line spanned by an eigenvector:

stabilizing a line spanned by \( v \) means:
\[ A v \parallel v \quad \text{(i)} \]
\[ A v = \lambda v \quad \text{for some scalar } \lambda. \quad \text{(ii)} \]

our def. of “eigenvector”.

The point is: eigenvectors (if you can find them) are a more natural basis for our linear op.