Today : 1) isometries (orthogonal transformations) (8.3)

2) Eigenvalues and eigenvectors (9.1–9.2)

Recall : last time we defined isometries :

$A : V \to W$ is an isometry if

$\forall\, x, y \in V$, $(Ax, Ay) = (x, y)$

inner product in W — inner product in V.

Today : 1) some properties of isometries.

① "Isometry" means: preserves length

Prop : A is an isometry $\iff$ $\forall v \in V$ $\|v\|_V = \|Av\|_W$

Proof : think of $(v+w, v+w) = \|v+w\|^2$

we can write it as,

$$\|v+w\|^2 = (v+w, v+w) = \underset{\|v\|^2}{(v,v)} + 2(v,w) + (w,w)$$
$$= \|v\|^2 + 2(v,w) + \|w\|^2$$

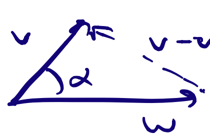
If A preserves lengths, then $\|A(v+w)\|^2$

$$= \|v+w\|^2$$
$$= \|Av\|^2 + 2(Av, Aw) + \|Aw\|^2$$

$$= \|v\|^2 + 2(Av, Aw) + \|w\|^2$$

and we get that $2(Av, Aw) = 2(v, w)$.

Picture :



$\leftarrow$ If $\|v\|$ and $\|w\|$ are preserved and the length of the third side is preserved, the angle α must be preserved, too. ∎

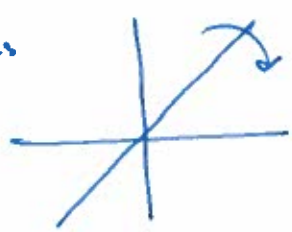
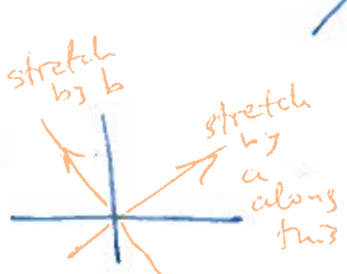
Def: A square matrix A is called orthogonal if $AA^t = Id$

Proposition Let V be a Euclidean space. Let $\{v_1, \dots, v_n\}$ be an orthonormal basis in V. Then $A : V \to V$ is an isometry $\iff$ the matrix of A with respect to this basis is orthogonal.

Pf: It is part of the theorem below the example.

Example isometries in $\mathbb{R}^2$: (usual dot product)

recall: bijective rtle linear transf : $\mathbb{R}^2 \to \mathbb{R}^2$

- Id – an isometry.
- rotations  – isometries
- reflections  – isometries
- 

dilations – not isometries
- **shear** transf. ← see the book (or not) – not isometries

• We will be able to classify all lin. transf. : $\mathbb{R}^2 \to \mathbb{R}^2$ (and even generally : $F^n \to F^n$) ← without complete proof

Prop : $A: V \to V$. A is orthogonal (i.e. an isometry) $\Rightarrow$ A is bijective ($\ker(A) = \{0\}$ and $rk(A) = \dim V$)

Pf : $(Av, Av) = (v, v)$

So $Av = 0 \Rightarrow (Av, Av) = 0 \Rightarrow (v, v) = 0$
$\Rightarrow v = 0$.

So $\ker(A) = \{0\}$. Then $rk(A) = \dim V$.

Theorem : The following conditions are equivalent :

① $A: V \to V$ is an isometry

② A takes an orthonormal basis to an orthonormal basis

Now let A also be the matrix of this linear transf. with respect to an orthonormal basis

④ $AA^t = Id$

⑤ $A^t A = Id$

⑥ The columns of A form an orthonormal basis

⑦ The rows of A form an orthonormal basis

Sketch of the proof (1) $\Rightarrow$ (2) - by definition

(2) $\Rightarrow$ (1) : Suppose $\{e_1, \dots, e_n\}$ is an orthonormal basis.

we want to prove: $(v,w) = (Av, Aw)$
$\forall\ v, w \in V$

let $v = \sum_{i=1}^{n} a_i e_i$

$w = \sum b_i e_i$

Then $(v,w) = \sum_{i=1}^{n} a_i b_i$ (because the basis is orthonormal)

Then $(Av, Aw) = \left(\sum_{i=1}^{n} a_i A(e_i), \sum_{i=1}^{n} b_i A(e_i) \right)$

$$= \sum_{j,i=1}^{n} a_i b_j \underbrace{(A(e_i), A(e_j))}_{= \delta_{ij}} = \sum_{i=1}^{n} a_i b_i = (v,w)$$

$\leftarrow$ by (2)

(2) $\Leftrightarrow$ (6) – by definition of a matrix of a linear transf. (the columns of A are the images of the basis vectors)

(6) $\Leftrightarrow$ (5) : columns of A are rows of A^t.

Thus, $$A^t A = \underbrace{\begin{bmatrix} \text{———} \\ \text{ith colum of } A \\ \text{———} \end{bmatrix}}_{A^t} \underbrace{\begin{bmatrix} & | & | & \\ & | & | & \\ & | & | & \end{bmatrix}}_{A}$$

j-th col. of A

$= [\delta_{ij}] = Id$

because colums of A are orthogonal and of length 1.

(5) $\Leftrightarrow$ (4) by taking transposes

(7) $\Leftrightarrow$ (6) : A is orthogonal $\Leftrightarrow$ A^t is orthogonal (by (4) and (5))

so we can apply our statement to A^t, and get that A^t has orthogonal columns $\Leftrightarrow$ A is an isometry.

But the columns of A^t are the rows of A.

<u>Corollaries</u> (1) If A is an orthogonal matrix, then $\det(A) = \pm 1$.

<u>Note</u>: If $\det(A) = 1$, A is called <u>special orthogonal</u>

<u>Notation</u> $O_n(\mathbb{R}) = \{ A \in M_n(\mathbb{R}) : A \text{ is orthogonal} \}$

– the <u>group</u> of orthogonal matrices

$SO_n(\mathbb{R})$ – the group of <u>special</u> orthogonal matrices.

(see § 8.4. – <u>optional</u> reading)

(2) A is orthogonal $\Leftrightarrow$ $A^{-1} = A^t$
(This is because $AA^t = Id$)

[<u>Note</u>: this means an interesting thing about minors of an orthogonal matrix: remember the formula for A^{-1} using the adjugate]

② Next topic: eigenvalues
eigenvectors
"spectral theory" -
the study of eigenvalues/eigenvectors.

Applications: everywhere!
Differential equations
quantum mechanics
signal processing
Fourier transform
computer science, AI, ...

Suppose $A: V \to V$ - given lin. op. (from V to itself)

Def: A vector $v \in V$ is called an eigenvector for A with eigenvalue $\lambda \in F$ if $Av = \lambda v$. $\boxed{v \neq 0}$

Will prove next class that they exist if $F = \mathbb{C}$.
Over arbitrary fields, ~~they~~ a given lin. op. might not have eigenvalues in that field.

Our goal: figure out how to find them when they exist.
And use them.

Example: for $A = \lambda \cdot Id$, every vector in V is an eigenvector with eigenvalue λ.

Main observation: if $v \in V$ is an eigenvector for A with eigenvalue λ, then

$$v \in \ker (A - \lambda Id)$$

indeed, if $Av = \lambda v$, then $Av - \lambda v = 0$.

$$Av - \lambda v = (A - \lambda \cdot Id) v$$

Then $\det(A - \lambda Id) = 0$
(write the matrix for $A - \lambda Id$ in any basis
(det does not depend on the choice of basis.)

Then the way to find eigenvalues is solve the equation $\det(A - \lambda Id) = 0$ for λ.

Example let $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$ ← mean, the lin. op. given by this matrix in the standard basis of $\mathbb{C}^2$.

Let's find its eigenvalues and eigenvectors.

The matrix of $A - \lambda \mathrm{Id}$ (w.r.t. the standard basis)

is $\begin{bmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{bmatrix}$

$$\underbrace{\det \begin{bmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{bmatrix}}_{\text{quadratic polynomial in } \lambda} = (1-\lambda)^2 - 4 = 1 - 2\lambda + \lambda^2 - 4$$
$$= \lambda^2 - 2\lambda - 3.$$

We want: the values of λ for which it is 0!

$$\lambda_{1,2} = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm 4}{2} = -1 \text{ or } 3.$$

(This is why we are working over $\mathbb{C}$: over $\mathbb{R}$, we don't have to have roots!)

This one has real roots.

How to find eigenvectors:

(The only way to solve these questions: find eigenvalues as roots of $\det(A - \lambda I)$, then solve for the eigenvectors)

Now that we know the eigenvalues, we can make a system of equations:

for λ_1: $Av = \lambda_1 v$, or better: $(A - \lambda I)v = 0$.

$\lambda_1 = -1$:

$$\begin{bmatrix} 1-(-1) & 4 \\ 1 & 1-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

solve as usual. : $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

it should have infinitely many solutions!

(if it doesn't, your "eigenvalue" is wrong).

$x_1 + 2x_2 = 0$. so $x_1 = -2x_2$.

Pick a vector that spans $\ker(A-\lambda \mathrm{Id})$.

Here, $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ works.

(<u>Note</u>: if v is an eigenvector for A with eigenvalue λ, then $a \cdot v$ with $a \neq 0$ also satisfies this).

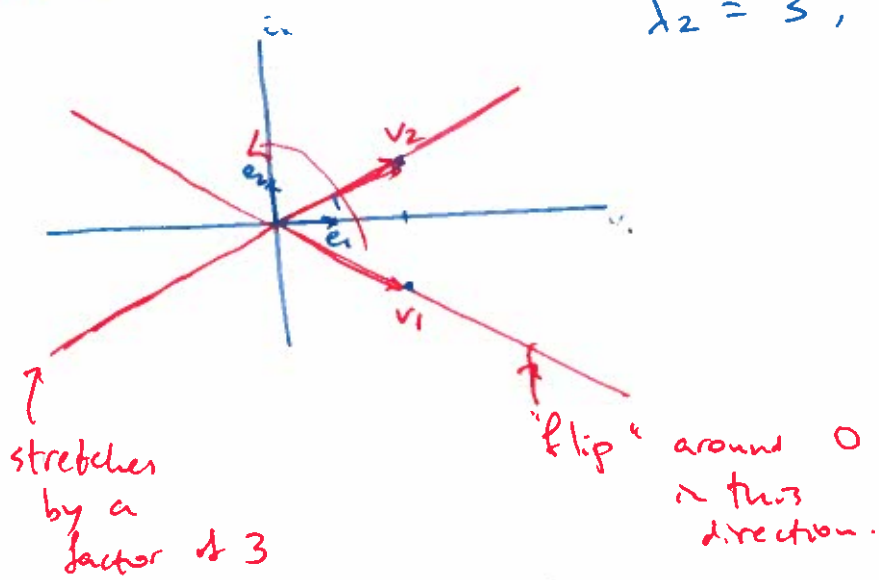
For $\underset{3}{\underset{\shortparallel}{\lambda_2}}$: $\begin{bmatrix} 1-3 & 4 \\ 1 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 - 2x_2 = 0 \qquad x_1 = 2x_2$

get: $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = v_2$.

So: for our A, $\lambda_1 = -1$, eigenvector is $\begin{bmatrix} -2 \\ 1 \end{bmatrix} = v_1$

$\lambda_2 = 3$, eigenvector is $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = v_2$



The matrix for A in the basis $\{v_1, v_2\}$ is $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ - diagonal matrix

eigenvalues on the diagonal.

$$\boxed{\text{So our } A \text{ \underline{has} an eigen\underline{basis}}}$$

<u>Next time</u>: in general.

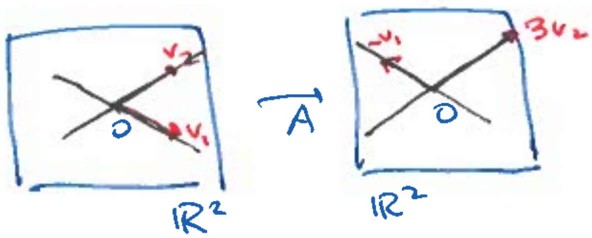
Recall: Example
$A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$: we figured out: $\lambda_1 = -1$, $\lambda_2 = 3$ - eigenvalu

with eigenvectors $v_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

determined up to a scalar

What are eigenvectors geometrically?

We had $A : \mathbb{R}^2 \to \mathbb{R}^2$



in the $\{v_1, v_2\}$ basis, A has matrix $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$

eigenvalues on the diagonal

How do we know A? - we could try to say what it does "rotation, stretch.., shear..."

or we give it by a matrix in the standard basis (this is what we did).

- Does A stabilize any line?
  the line doesn't move.

Any such line is a line spanned by an eigenvector:
stabilizing a line spanned by v means: $Av \parallel v$

$\iff Av = \lambda v$ for some scalar λ.

our def. of "eigenvector".

The point is: eigenvectors (if you can find them) are a more natural basis for our linear op.