<u>Today</u>: · Eigenvalues
· char. polynomials
· Fundamental Theorem of Algebra

<u>Recall from last time</u> $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$

eigenvalues: $-1, 3$ –

roots of the <u>characteristic polynomial</u>

$$p_A(\lambda) = \det(A - \lambda \, Id) = \det\left(\begin{bmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{bmatrix}\right)$$

<u>eigenvectors</u>: $v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $(\lambda = -1)$ $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $(\lambda = 3)$

<u>Some theory</u>:

① · Why is $\det(A - \lambda I)$ a polynomial in λ?
Does it always have roots?

② · When does A have a basis of eigenvectors?

③ · How to switch between bases?

① $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ $A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ a_{n1} & \cdots & & a_{nn}-\lambda \end{bmatrix}$

$$\det(A - \lambda I) = \underbrace{(a_{11} - \lambda)\begin{vmatrix} & \\ & \end{vmatrix}}_{\text{deg } n-1 \text{ poly in } \lambda \text{ by ind. assumption.}} - a_{12}\begin{vmatrix} a_{21} & \cdots & a_{2n} \\ & \ddots & \\ & & \end{vmatrix}$$

deg n in λ

similar thing $(n-1)\times(n-1)$

deg $\le n-2$ by ind. assump.

one term contains λ in each row/column

Statement: If we have a determinant s.t. every row and column contains only one linear term in the variable λ, then such a $n\times n$ determinant is a polynomial of deg n in λ.

base: $n=1$: $|a_{11}-\lambda| \to$ deg 1 in λ.

Induction step: use cofactor expansion as above.
Get a sum of terms, one of them is degree n in λ, the others are lower degree.

Note! Using Leibniz rule, it is easy to see this without induction.

Fundamental Theorem of Algebra (proved, e.g., using complex analysis)

The field of complex numbers $\mathbb{C}$ is algebraically closed, which means: every polynomial with coeffs in $\mathbb{C}$ has a root in $\mathbb{C}$.

Ex: in $\mathbb{R}$, this is not so: x^2+1 has no real root.

Magic: we made $\mathbb{C}$ by putting in" a root of x^2+1.

Corollary: Over $\mathbb{C}$, every polynomial factors as a product of linear factors:

$$p(x) = \underset{\substack{\uparrow \\ \text{constant}}}{a_n}(x-\alpha_1)(x-\alpha_2)\dots\underset{\substack{\uparrow \\ \text{roots}}}{(x-\alpha_n)}$$

$$p = a_n x^n + \dots + a_0$$

A polynomial of degree n has exactly n roots in $\mathbb{C}$

Pf: a is a root of $p(x) \iff p(x) = (x-a)f(x)$ for some polynomial $f(x)$.

(Bezout's thm). So, use this + induction.

Def: If $p(x) = (x-\alpha)^k f(x)$, k is called the multiplicity of α

Example $p(x) = (x-1)^3 x^2 (x+3)^{10}$

Then 1 is a root of multiplicity 1,

0 — , — 2

−3 — , — 10.

For the eigenvalues (i.e., roots of the characteristic polynomial) these multiplicities are called algebraic multiplicities.

Comment: for $n \geq 5$, there is no formula for finding the roots! (Galois)

In practice, this means we cannot expect to find eigenvalues exactly. Numerically, can find eigenvalues approximately.

Def: If $p(x) = a(x-\alpha)^k \cdot h(x)$ the greatest such k is called the multiplicity of the root α.

- If start with an $n \times n$ matrix A, write its characteristic polynomial $p_A(\lambda) = \det(A - \lambda I)$

If $p_A(\lambda)$ has n distinct roots in $\mathbb{C}$ (meaning every root has multiplicity $=1$)

then A will have a basis of eigenvectors

Will prove this next class.

If there are multiple roots, it is more complicated.

- there might still be a basis of eigenvectors, or not. Not easy to distinguish these cases.

What happens over $\mathbb{R}$?

1) Let $p(x)$ be a polynomial with real coefficients. Then $p(x)$ has to factor into linear factors over $\mathbb{C}$.

Note: if we apply complex conjugation, the real coefficients of p do not change, i.e. p does not change! But its roots get conjugated. This means, the roots of $p(x)$ come in pairs of complex conjugate ones.

Suppose $\alpha = a+bi$ is a root of $p(x)$. Then $\bar{\alpha} = a-bi$ is also a root. Consider the corresponding linear factors together: $(t-\alpha)(t-\bar{\alpha})$

$$= (t-(a+bi))(t-(a-bi))$$

$$= \underbrace{t^2 - 2at + (a^2+b^2)}_{\text{irreducible quadratic polynomial with real coefficients.}}$$

We get that, a polynomial with real coefficients must factor like this:

$$p(x) = (x-\lambda_1)^{a_1} \cdots (x-\lambda_k)^{a_k} \cdot \prod_{j=1}^{m} \left(x^2 - 2\alpha_j x + (\alpha_j^2+\beta_j^2)\right)^{a_j}$$

linear factors corresp. to real roots

quadratic factors corresp. to complex roots

<u>Corollary</u> A polynomial of <u>odd</u> degree with real coeffs must have a real root (in fact, an <u>odd</u> number of real roots)

<u>What does this mean for matrices?</u>

<u>Example</u> Consider $A: \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

Its eigenvalues:

$$\det \begin{bmatrix} a-\lambda & b \\ -b & a-\lambda \end{bmatrix} = 0$$

$$\Leftrightarrow \quad (a-\lambda)^2 + b^2 = 0$$

$$\Leftrightarrow \quad a-\lambda = \pm bi \quad \Leftrightarrow \quad \boxed{\lambda = a \pm bi}$$

<u>What does this linear operator do?</u>

Rewrite this matrix as:

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \sqrt{a^2+b^2} \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \\ -\frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{bmatrix}$$

There exists an angle φ s.t.

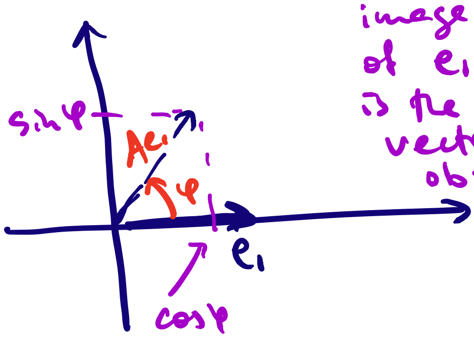
$$\cos\varphi = \frac{a}{\sqrt{a^2+b^2}}, \quad \sin\varphi = \frac{-b}{\sqrt{a^2+b^2}}$$

Then our matrix is:

$$\sqrt{a^2+b^2}\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

image of e_1 is the vector obtained by rotating e_1 by φ.

rotation of e_2 by φ.



So A rotates by the angle φ and scales by the real number $\sqrt{a^2+b^2}$.

Why does this make sense: of course a rotation does not have real eigenvalues

– it does not leave any line in its place, so it cannot have a real eigenvector.

What we just saw is an example of a

real matrix that did not have real eigenvalues, and it turned out to be a rotation. We will see that in fact <u>all</u> real 2×2 -matrices without real eigenvalues are rotations.

<u>Side comment</u>: if v is an eigenvector for A with eigenvalue λ, then on the line spanned by v, A acts just by <u>stretching</u> by λ (if $\lambda \in \mathbb{R}$).

Now, if $\lambda \in \mathbb{C}$, there is no <u>real</u> <u>line</u> for λ, but there is a plane $\mathbb{R}^2$.

On this plane, as we just saw, A acts by rotation and scaling.

But notice that if $\lambda = a + bi$, it can be written as:

$\lambda = \sqrt{a^2+b^2}\, e^{2\pi i \varphi}$, where φ is the same angle as above.

Rotation by φ is nothing but multiplication (in $\mathbb{C}$) by $e^{2\pi i \varphi}$

So if we identity this $\mathbb{R}^2$ with $\mathbb{C}$, we can still think that A acts on it by multiplication by the complex number λ, so it is not a mystery that we see the rotation.

Summary : 1) to find eigenvalues for $A : V \to V$, we write its characteristic polynomial

$p_A(\lambda) = \det (A - \lambda \mathrm{Id})$, and factor it somehow.

2) If we managed to factor it into linear factors, we can start looking for eigenvectors for each root, solving systems of linear equations

3) Over $\mathbb{C}$, any polynomial will factor.

4) over $\mathbb{R}$, we might get some real roots, and some irreducible quadratic factors, corresponding to pairs of complex conjugate roots. The quadratic factors mean that A acts by rotation on some

corresponding planes.

<u>Next class</u>: Conditions for existence of a basis of eigenvectors.