Today: • When does a basis of eigenvectors exist?
(a sufficient condition) 9.2

• How to go between different bases
(equivalence of matrices) → 11.1, 11.2, some 11.4

Recall: Last time we finished at the Fundam. Thm of Algebra:

every polynomial over $\mathbb{C}$ factors into a product of linear factors:

$$p(x) = a_n (x-\alpha_1) \cdots (x-\alpha_n) \quad (n = \deg(p)) \quad (\text{see } 9.4)$$

roots.

Comment: for $n \geq 5$, there is no formula for finding the roots! (Galois)

In practice, this means we cannot expect to find eigenvalues exactly. Numerically, can find eigenvalues approximately.

Def: If $p(x) = a(x-\alpha)^k \cdot h(x)$
the greatest such $k$ is called the multiplicity of the root $\alpha$.

• If start with an $n \times n$ matrix $A$,
write its characteristic polynomial $p_A(\lambda) = \det(A - \lambda I)$

If $p_A(\lambda)$ has $n$ distinct roots in $\mathbb{C}$
(meaning every root has multiplicity $=1$)

then $A$ will have a basis of eigenvectors

If there are multiple roots, it is more complicated.
- there might still be a basis of eigenvectors, or not. Not easy to distinguish these cases.

Recall why eigenvalues are roots of $p_A(\lambda)$:
because $\ker(A-\lambda I) \neq \{0\}$ when $\lambda$ is an eigenvalue

Def: • $\dim \ker(A-\lambda I)$ is called the geometric multiplicity of the eigenvalue $\lambda$.

It is not 1 only if $\lambda$ is also a multiple root of $p_A(\lambda)$ and in this case, the multiplicity of $\lambda$ as a root of the char. poly. $p_A(\lambda)$ is called algebraic multiplicity.

Caution: geom. multiplicity $\leq$ alg. multiplicity
but they don't have to be equal!

To find geom. mult., you have to: do row reductions for $A-\lambda I$, find its kernel.

There is a method for finding a particularly good basis for this kernel, but it's not part of Math 223
And then need more basis vectors for V

**Thm** If $\underbrace{v_1^{(1)}, \dots, v_{k_1}^{(1)}}$ are lin. indep. eigenvectors for eigenvalue $\underline{\lambda_1}$,

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$v_1^{(m)} \dots v_{k_m}^{(m)}$ are lin. indep. eigenvectors for the eigenvalue $\lambda_m$,

then $\{v_1^{(1)}, \dots v_{k_m}^{(m)}\}$ are lin. indep.

all of them

(eigenvectors corr. to different eigenvalues are lin. indep.)

based on:

**Lemma**: If $\lambda_1 \dots \lambda_k$ are eigenvalues of $A: V \to V$, $\lambda_i \neq \lambda_j$ for $i \neq j$

Let $v_i$ be an eigenvector for $\lambda_i$

then $v_1, \dots, v_k$ are lin. indep.

**Pf**: Induction on $k$.

**Base**: $\underline{k=1}$: $v_1$ - eigenvector for $\lambda_1$

$v_1 \neq 0$ by def'n.

Nothing to prove.

**Induction step**: Suppose we know this statement for $k=n$. Want to prove it for $k=n+1$.

Let $v_1 \dots v_{n+1}$ be eigenvectors for $\lambda_1, \dots, \lambda_{n+1}$.

Suppose we have

$$(1) \quad \boxed{\alpha_1 v_1 + \dots + \alpha_n v_n + \alpha_{n+1} v_{n+1} = 0}$$

for some $\alpha_1, \dots, \alpha_{n+1} \in F$.

our field

Want to prove:

all $\alpha_i = 0$

Apply $A$: Get.

$$A(\alpha_1 v_1 + \dots + \alpha_{n+1} v_{n+1}) = 0.$$

$$= \alpha_1 A \cdot v_1 + \dots + \alpha_{n+1} A v_{n+1}$$

$$= \boxed{\alpha_1 \lambda_1 v_1 + \dots + \alpha_{n+1} \lambda_{n+1} v_{n+1} = 0.} \quad (2)$$

b/c eigenvectors.

Subtract $\boxed{\lambda_{n+1}\cdot(1)}$ from $\boxed{(2)}$:

Get: (the last term cancels!)

$$(\alpha_1\lambda_1 v_1 - \alpha_1\lambda_{n+1} v_1) + \cdots + (\alpha_n\lambda_n v_n - \lambda_{n+1}\alpha_n v_n) = 0$$

$$\Downarrow \quad \alpha_1\underbrace{(\lambda_1 - \lambda_{n+1})}_{\neq 0} v_1 + \cdots + \alpha_n\underbrace{(\lambda_n - \lambda_{n+1})}_{\neq 0} v_n = 0.$$

By induction assumption, $\{v_i\}$ are lin. indep.

And it was given that $\lambda_i \neq \lambda_j$ for $i \neq j$

so $\lambda_i \neq \lambda_{n+1}$ when $i = 1, \ldots, n$.

Then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

Thus we have $\alpha_{n+1} v_{n+1} = 0$. So $\alpha_{n+1} = 0$.

(b/c $v_{n+1} \neq 0$ by def. of an eigenvector)

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**Corollary:** If the roots of the char. poly (over $\mathbb{C}$) are distinct, the lin. op. has a basis of eigenvectors

Pf: You have $n$ eigenvalues, their eigenvectors are lin. independent.

$n = \dim(V)$

since there's $n$ of them, they form a basis..

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The matrix of $A$ in this basis is diagonal!

How to change bases:

# Changes of basis

- What happens to coordinates
- What happens to matrices

when you change the basis?

Let $v_1^{old}, \dots v_n^{old}$ be a basis in $V$

And we take another basis, call them $v_1^{new} \dots v_n^{new}$.

$x \in V$ $\quad x = x_1^{old} v_1^{old} + - + x_n^{old} v_n^{old}$

$$= \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^{old} \text{ in the old basis.}$$

$$x = x_1^{new} v_1^{new} + - - + x_n^{new} v_n^{new}$$

$$= \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^{new} \text{ in the new basis.}$$

Make the "transition matrix" or "change of basis" matrix

$$C = \begin{bmatrix} c_{11} & & \\ \vdots & \cdots & \\ c_{1n} & & \end{bmatrix} \qquad v_i^{new} = \begin{bmatrix} c_{1i} \\ \vdots \\ c_{ni} \end{bmatrix} \leftarrow \text{coords in the } \underline{\text{old}} \text{ basis}$$

$v_1^{new}$ columns are the coordinates of the new basis vectors w.r.t. the old basis.

we have: $\boxed{x^{old} = C x^{new}}$ ← easier to see

$\boxed{x^{new} = C^{-1} x^{old}}$

We proved it earlier in the course. It is easy to see because this relation holds for the new basis vectors, and then it holds for all vectors b linearity.

## For linear operators

$A : V \to V$ - lin. op.

has matrix $A^{old}$ w.r.t. the old basis $\{v_1 \dots v_n\}^{old}$

$A^{new}$ — " — $\{v_1 \dots v_n\}^{new}$.

How are these matrices related?

$x^{new} = C^{-1} x^{old}$

$$\underbrace{(A^{old} x^{old})}_{\in V}^{new} = A^{new} x^{new}$$

both are expressions for $Ax$ — the same vector, expressed in the new coordinates.

lin. operator: $Ax$

$(A x^{old})^{new}$ makes also no sense

$x \in V \to Ax \in V$

old coord, new coords; old, new coord.

Then: $C^{-1} A^{old} x^{old} = A^{new} x^{new}$

$C^{-1} A^{old} C x^{new} = A^{new} x^{new}$

$$\boxed{A^{new} = C^{-1} A^{old} C}$$

Example: $C = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$

recall: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ were eigenvectors for $A$

compute $C^{-1} A C$: $\quad C^{-1} = \begin{bmatrix} -1 & -2 \\ -1 & 2 \end{bmatrix} \cdot \frac{1}{-4}$ "det A"

using $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $= \begin{bmatrix} 1/4 & 1/2 \\ 1/4 & -1/2 \end{bmatrix}$

$$C^{-1} \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} C = \begin{bmatrix} 1/4 & 1/2 \\ 1/4 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} \leftarrow A^{new}$$

$\boxed{\text{it does}}$ → expect $= \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

## An illustration for Changes of basis

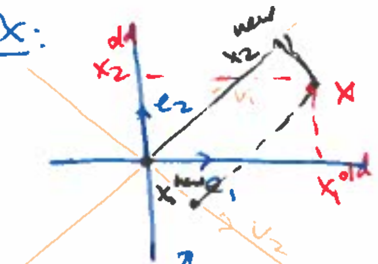
$V$ - vector space over $F$.

$\{v_1^{old}, \dots, v_n^{old}\}$ - a basis of $V$

$\{v_1^{new}, \dots, v_n^{new}\}$ - a new basis of $V$.

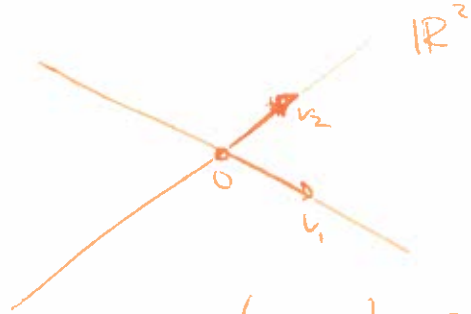
(Recall: they have the same number of elements, $n = \dim(V)$).

Ex:




"old" basis

$\mathbb{R}^2$




'new basis.

(often: old basis is $\{e_1, \dots, e_n\}$ - standard basis of $F^n$)

How do the old coords of $x$ relate to the new coords of $x$?

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^{old} = x_1^{old} v_1^{old} + \dots + x_n^{old} v_n^{old}$$

Also

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^{new} = x_1^{new} v_1^{new} + \dots + x_n^{new} v_n^{new}$$

Geometrically hard to relate $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^{new}$ to $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^{old}$.

Above, we solved this question algebraically:

we make a transition matrix $C$:

Columns of $C$ are the coordinates of the new basis vectors in the old basis.

Example: Old basis: $\{e_1, e_2\}$ in $\mathbb{R}^2$

New basis: $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^{old}$ What are the new coordinates?

↑ usual standard coords.

We make $C: \begin{array}{cc} \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow \\ v_1 & v_2 \end{matrix} \end{array}$ - our transition matrix

Aside: you could make a mean example:

$\{v_1 \ldots v_n\}$ - not the standard basis, given in terms of $e_1 \ldots e_n$

$\{w_1 \ldots w_n\}$ - another one. - given in terms of $e_1 \ldots e_n$.

Transition from $\{v_1 \ldots v_n\}$ to $\{w_1 \ldots w_n\}$?

Answer: $C_1$ = transition matrix from $\{e_1, \ldots, e_n\}$ to $\{v_1, \ldots, v_n\}$

$C_2$ - transition from $\{e_1, \ldots, e_n\}$ to $\{w_1, \ldots, w_n\}$

$C_1 = \begin{bmatrix} \boxed{\phantom{x}} & & \end{bmatrix}$ $C_2 = \begin{bmatrix} \boxed{\phantom{x}} & & \end{bmatrix}$

coords of $v_i$ w.r.t. $\{e_1, \ldots, e_n\}$

↑ coords of $w_i$ w.r.t. $\{e_1, \ldots, e_n\}$

To go from the $v$'s to the $w$'s:

$\boxed{C^{-1} = C_2^{-1} C_1}$ $\rightsquigarrow$ $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^{new, v} = C_1^{-1} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^{standard}$

$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^{w} = C_2^{-1} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^{standard}$

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We have: $x^{old} = C \cdot x^{new}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C \cdot \begin{bmatrix} x_1^{new} \\ x_2^{new} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1^{new} \\ x_2^{new} \end{bmatrix}$$

Then $\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^{new} = C^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^{old}}$

why does $C^{-1}$ exist?

We know: columns of $C$ form a basis, so $rk(C) = n$
then $C^{-1}$ exists (b/c $\dim(\ker(C)) = n - rk(C) = 0$.)