<u>Last time</u>: We proved a number of very important things: let $A: V \to V$

- The <u>eigenvalues</u> of a matrix A are roots of its char. poly $p_A(\lambda) = \det(A - \lambda I)$

- If the roots are distinct, then V has a basis that consists of <u>eigenvectors</u> of A.

- This means, in this new basis of eigenvectors, the matrix for A is diagonal.

In terms of matrices, this means:
if A is the matrix of our linear operator with respect to the <u>standard</u> basis,
then we can form the <u>transition matrix</u> C

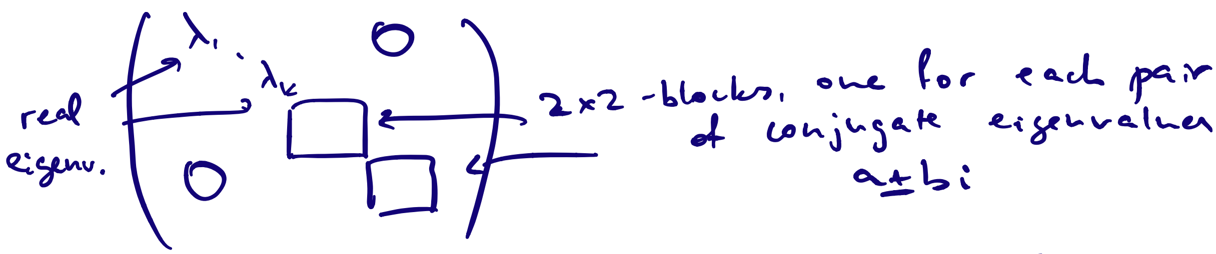
Its <u>columns</u> are the coordinates of the eigenvectors of A with respect to the standard basis.

Then we have $C^{-1} A C = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ – a diagonal matrix with the eigenvalues of A on the diagonal.

Now, what happens if A doesn't have n distinct eigenvalues?

* If A is a real matrix, then its eigenvalues come in pairs of complex conjugates.
  To each pair corresponds a plane in our space $\mathbb{R}^n$ on which A acts via a composition of a rotation and a dilation.

  More precisely, there exists a basis for $\mathbb{R}^n$ in which the matrix of A is block-diagonal, with blocks of the form:



The blocks are of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

$$= \sqrt{a^2+b^2} \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}_{\text{rotation}}$$

$$\boxed{\begin{array}{l} \text{here} \\ \lambda = a \pm bi \\ \quad = \sqrt{a^2+b^2}\, e^{i\varphi} \end{array}}$$

(This is still under the condition that the eigenvalues are <u>distinct</u>)

(This part about real 2×2-blocks will not be on exams).

Question: How does this relate to the matter of A being injective/surjective?

Here $A : V \to V$ - from a space to itself.
Recall that then the following conditions are equivalent:

1) A is injective
2) A is surjective
3) A^{-1} exists
4) $\det(A) \neq 0$.

Now we can add one more condition equivalent to the above: 0 is not an eigenvalue of A.

Indeed, 0 is an eigenvalue $\iff \ker(A) \neq 0$
(exists $v \neq 0$ s.t. $Av = 0 \cdot v$)

You can also see this using the det:
$\det(A) = 0 \iff \det(A - 0 \cdot Id) = 0$.

So the question of whether the eigenvalues are <u>distinct</u> is <u>not</u> about injectivity/surjectivity.

What happens when the eigenvalues are <u>not</u> distinct?

There is a number of possibilities.
Suppose λ is a root of $p_A(t)$ of multiplicity m:
$p_A(t) = (t-\lambda)^m \, p_1(t) \leftarrow p_1(t)$ has no factor $t-\lambda$.

Consider two main examples

Example 1 $\quad A = \begin{pmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \lambda \end{pmatrix} = \lambda \cdot \mathrm{Id}.$

Then $\quad p_A(t) = (\lambda - t)^m$

Every vector is an eigenvector for $\lambda \cdot \mathrm{Id}$!

Example 2 $\quad$ Let $J_\lambda = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$

The char. poly is the same: $(\lambda - t)^3$.

But there is only one eigenvector:

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

has only one solution: $z = y = 0$ (up to scaling)

$$\Updownarrow$$

$$\begin{cases} \lambda x + y = \lambda x \\ \lambda y + z = \lambda y \\ \lambda z = \lambda z \end{cases}$$

$(1, 0, 0)$ is the only eigenvector (up to scaling)

In other terms, here $\boxed{\dim \ker (J_\lambda - \lambda \cdot \mathrm{Id}) = 1}$

Def: $\dim \ker (A - \lambda \mathrm{Id})$ is called the geometric multiplicity of λ.

It is always $\leq$ the algebraic multiplicity, but as we see, can be strictly smaller.

Jordan normal form

How much can we simplify a matrix up to a change of basis?

- If the characteristic polynomial of A has n distinct roots

$$p_A(\lambda) = \det(A - \lambda I)$$

then A has a basis of eigenvectors

In this basis, A is diagonal!

(for each v_i, $Av_i = \lambda_i v_i$, so the ith column of A is $\begin{bmatrix} 0 \\ 0 \\ \lambda_i \\ 0 \\ 0 \end{bmatrix} \leftarrow$ ith place)

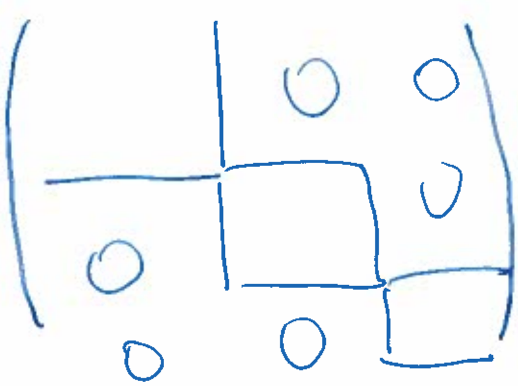
in the eigenvector basis, the matrix of A

is $\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$.

- If $p_A(\lambda)$ has multiple roots: $p_A(\lambda) = (x-\lambda_1)^{k_1} \cdots (x-\lambda_n)$

then cannot always make it diagonal.

But can get it to Jordan Normal Form (cannot prove)



each block looks like:

$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \lambda & 1 \\ 0 & & & \lambda \end{pmatrix} \leftarrow J_n(\lambda) \text{ - Jordan block}$$

the same eigenvalue

For a multiple eigenvalue, get smth like:

$$\begin{pmatrix} \boxed{\begin{matrix}\lambda & 1 & \\ & \lambda & 1 \\ & & \lambda\end{matrix}} & 0 & \\ & \boxed{\lambda} & 0 \\ & & \boxed{\lambda} \end{pmatrix}$$

size 3 block

size 1 blocks

λ has algebraic multiplicity 5.

$P_A(x)$ has root of mult. 5 at λ.

each block $\leftrightarrow$ one eigenvector.

here $\dim \ker(A-\lambda I)$ is 3 (the number of blocks)

Aside

not part of this course

Hard part: dealing with the situation when we do not have "enough" eigenvectors: when geometric multiplicity $\dim\ker(A-\lambda I)$ is smaller than algebraic multiplicity of λ.

Then we need to find more basis vectors $v_1, v_2, \ldots$ such that:

$(A-\lambda I)v_1 = v_0$ ← the eigenvector

$(A-\lambda I)v_2 = v_1$

This is hard because $A-\lambda I$ is not invertible! So hard to prove they exist; also they are not unique — how to make good choices?

This gives the Jordan block

$$\begin{array}{cccc} & v_0 & v_1 & v_2 \\ & \left(\begin{matrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{matrix}\right) \end{array}$$

here:

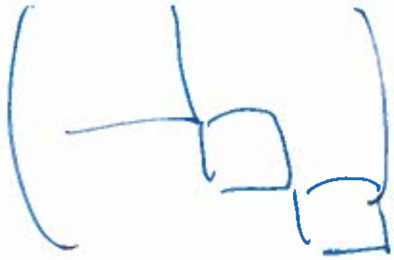
$Av_0 = \lambda v_0$

$Av_1 = \lambda v_1 + v_0$

$Av_2 = \lambda v_2 + v_1$

Last time: We stated the Jordan Normal Form Thm:

$A: V \to V$ there exists a basis for V s.t. the matrix of A looks like this:



block-diagonal

each block is $\begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ 0 & & & \lambda \end{pmatrix}$

(Note: $\boxed{\lambda} \leftarrow 1\times 1$-block)

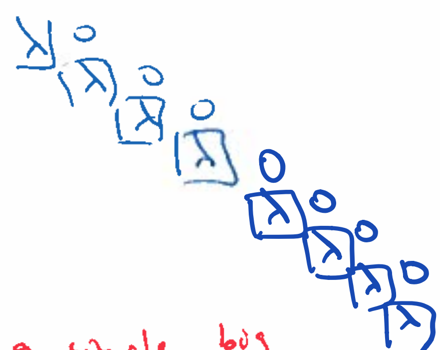
- Special case: $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix} \leftarrow$ if V has a basis of eigenvectors

  diagonal.

- Only multiple eigenvalues (roots of $p_A(\lambda) = \det(A - \lambda I)$ of multiplicity > 1 give rise to blocks)

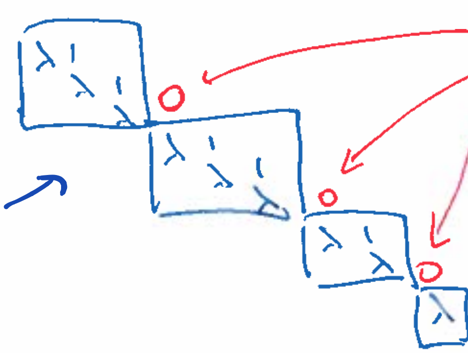
- We proved that if the eigenvalues are distinct, the matrix can be diagonalized.

- Warning: a repeated root of $p_A(\lambda)$ can give rise to several blocks: it can be:



or

Something like this



This is the difference between this and a single big block.

We will not do it.

(every partition of the algebraic multiplicity into the sum of the sizes of blocks is possible)

Side note: for a 2×2 Jordan block,

if λ is the only eigenvalue of $A: V \to V$, $\dim(V) = 2$,

then $\operatorname{Im}(A-\lambda I) \subset \ker(A-\lambda I)$

this was saying: for a 2-dim space, if λ has multiplicity 2, you get: $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

i.e. $A = \lambda I$

or $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$: $\ker(A-\lambda I)$ is 1-dim and coincides with $\operatorname{Im}(A-\lambda I)$

(i.e. $(A-\lambda I)^2 = 0$).

Reminder: change of basis $\longleftrightarrow$ $A^{new} = C^{-1} A^{old} C$

C = change of basis matrix.

Why do we care? "Matrix calculus"

- What if you wanted to apply the usual functions you know to matrices?
  - we know how to add and multiply matrices.

    note: $AB \neq BA$. A^{-1} doesn't always exist if $A \neq 0$.

- Especially, one wants to compute powers of a matrix (which corresponds to repeatedly applying the same linear operator).

- Want to <u>compute</u> A^m. (m some high power)

<u>Trick</u>: get A to Jordan normal form:

$$B = C^{-1} A C$$

block matrix (normal form of A)

Then $$B^m = \underbrace{(C^{-1}A\cancel{C})(\cancel{C}^{-1}AC)\ldots(C^{-1}AC)}_{m}$$

$$= C^{-1} A^m C$$

So $$A^m = \boxed{C B^m C^{-1}}$$

This is especially easy if B is diagonal:

$B = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ then $B^m = \begin{pmatrix} \lambda_1^m & & 0 \\ & \ddots & \\ 0 & & \lambda_n^m \end{pmatrix}$

In particular, the eigenvalues of A^m are $\lambda_1^m, \ldots, \lambda_n^m$!

<u>Aside</u>:
With blocks: also not bad:

$$\underset{k \times k\text{-block}}{\begin{pmatrix} \lambda & 1 & \\ & \ddots & 1 \\ 0 & & \lambda \end{pmatrix}} = \underset{D}{\begin{pmatrix} \lambda & & 0 \\ 0 & \lambda & \\ & & \lambda \end{pmatrix}} + \underset{N}{\begin{pmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ 0 & & 0 \end{pmatrix}}$$

commute!

$DN = ND$.

$N^k = 0$ (N is <u>nilpotent</u>)

$(D+N)^m$ = write the binomial formula, many terms disappear because of high powers of N)

- What other functions can we apply?

  Exponential!

  $$e^A = ??$$

  what is e^x? (as a fn of x).

  - a function s.t. $(e^x)' = e^x$

    ie. solution to the equation $f' = f$.

    (really, get a family of solutions, $c \cdot e^x$), choose one s.t. $f(0)=1$ so $c=1$

  (e is <u>defined</u> as its value at 1)

  - To compute: Taylor series:

  $$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots$$

  converges for all $x \in \mathbb{R}$

  (recall $n! = 1 \cdot 2 \cdot 3 \cdots n$)

  in fact, also converges for all $z \in \mathbb{C}$, defining e^z on $\mathbb{C}$

  (!) Can use this to <u>define</u>

  $$e^A = I_d + A + \frac{1}{2!}A^2 + \cdots + \frac{1}{n!}A^n + \cdots$$

  $Id = $ "1".

  If A is diagonal, $e^A = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}$

  (with some work, prove all converges...)

- All other functions that have convergent Taylor series — can be applied to matrices.

I did not say the next thing in class, but it gives a hint for the extras problem set, so I am including it in the notes. It is not going to be on exams.

An application
For diff. equations: $f'(x) = \lambda f(x) \rightsquigarrow f(x) = c \cdot e^{\lambda x}$

Now you can deal with systems of diff. eq:

$f_1, \dots, f_n$

$$\bar{f}(x) = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}' = A \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$

vector function

matrix of scalars

constant vector $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Solution: $\bar{f}(x) = e^{Ax} \cdot C$

In practice, we diagonalize A, get $e^{\lambda_1 x} \dots e^{\lambda_n x}$
get solutions of the form $\sum c_i e^{\lambda_i x}$

/ Multiple eigenvalues lead to terms of the form $x e^{\lambda x}$, $x^2 e^{\lambda x}$ -- /

Also: higher order diff. eq. can be converted to systems of diff. eq: $f'' = \lambda f$

trick: make $f_1 = f$
$f_2 = f'$

Then $f'' = \lambda f$ becomes $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}' = \begin{bmatrix} f_2 \\ \lambda f_1 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1 \\ \lambda & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$