Euclidean spaces

- Extra structure on a vector space V over ~~a field~~ $\mathbb{R}$.
  - dot product (or scalar product)

~~vector / affinity~~

Abstract definition

The dot product is a function ~~from~~ $(\,,\,) : V \times V \to \mathbb{R}$

satisfying:

1) **bilinear**:
   $\text{in } \mathbb{R} \leftarrow (x,y)$ is linear in x if y is fixed
   $\quad\quad\quad\text{in } V$ $\quad\quad\quad$ linear in y if x is fixed

   This means:
   $$(x_1 + x_2, \, y) = (x_1, y) + (x_2, y)$$
   $$(\lambda x, \, y) = \lambda (x, y) \quad \text{for} \quad \lambda \in \mathbb{R}$$
   $$(x, \, y_1 + y_2) = (x, y_1) + (x, y_2)$$
   $$(x, \, \lambda y) = \lambda (x, y) \quad\quad \text{for} \quad \lambda \in \mathbb{R}.$$

2) **symmetric** : $\quad (x,y) = (y,x)$

3) **positive-definite** : $\quad (x,x) \geqslant 0$ and $(x,x) > 0$ if $x \neq 0$.

   this is why we only work over $\mathbb{R}$.

Another term for a dot product:

"symmetric bilinear positive-definite <u>form</u>"

↑ function of 2 variables, linear in each.

Example: $x, y \in \mathbb{R}^n$
   Define $(x,y) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$
   - the usual dot product $x \cdot y$

In fact, there exist infinitely many different scalar products (symmetric bilinear pos. def. forms) on $\mathbb{R}^n$ (or on any real vector space).

Example: $A: \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear op.

need one more requirement: given by a symmetric matrix in the standard basis.

then $(x,y)_{new} = (Ax, Ay)$ — gives us a new scalar product.

Why we care: this gives us the notion of
- distance (or length), and
- angles.

Def: $\|x\| = \sqrt{(x,x)}$ ← The statement that for the usual dot product this agrees with the usual length is Pythagoras' theorem.

"the norm of x"

we could define the angle between x and y via its cosine:
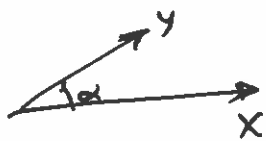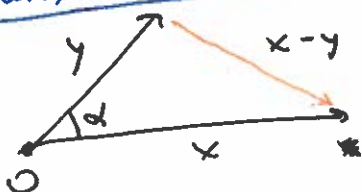

$$x \cdot y = \|x\| \|y\| \cos \alpha$$

so we define $\alpha = \cos^{-1} \left(\dfrac{(x,y)}{\sqrt{(x,x)} \sqrt{(y,y)}} \right)$

Something to prove before we can do it: (Cauchy-Schwarz inequality)

usual real abs. value

$$\left| \frac{(x,y)}{\sqrt{(x,x)} \sqrt{(y,y)}} \right| \leq 1. \quad (\text{in order to define } \cos^{-1}(\ldots))$$

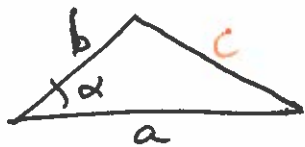next class, will prove it for the general inner product.

Let's compute $(x-y, x-y) = \|x-y\|^2$

$\|$

$(x,x) - (y,x) - (x,y) + (y,y)$

Get: $\boxed{\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2(x,y)}$

For the standard dot product, it agrees with the usual angles
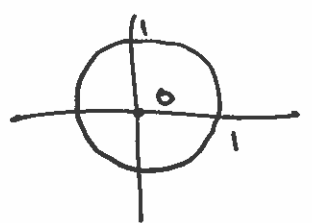
Law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos\alpha$$

$\underbrace{a^2}_{\|x\|^2}$ $\underbrace{b^2}_{\|y\|^2}$

So $(x,y) = \|x\|\,\|y\|\cos\alpha$.

we just proved that with the usual notion of length, the angle agrees with the formula we get from the dot product.

Example: Now that we have the norm, we have unit circle: $\{ x \in \mathbb{R}^2 : \|x\| = 1 \}$.

(in $\mathbb{R}^2$)

The usual dot product:

(our familiar unit circle)

If we use the new dot product to define angles, they will be different from our usual angles.

New dot product:

Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

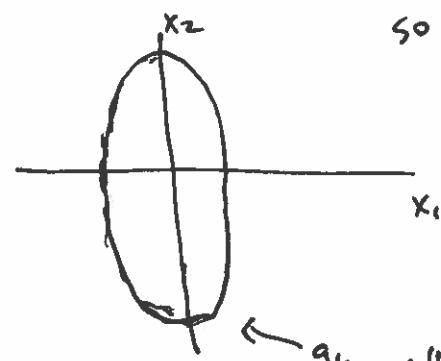stretches ✱ by a factor of 2 along x.

So $\|x\|_{new} =$

$\sqrt{(Ax, Ax)} = \sqrt{4x_1^2 + x_2^2}$

$x = (x_1, x_2)$

an ellipse

so the unit circle is:

$4x_1^2 + x_2^2 = 1$

↑

in the standard coords.

What we really care about, is the notion of orthogonality: x and y are orthogonal with respect to $(,)$ if $(x, y) = 0$.

Example of dot product in an infinite-dim. vector space

- Consider the space of all continuous functions
$$f: [-1,1] \to \mathbb{R}$$
  They form a real vector space. (infinite-dim'l)

Define
$$(f, g) = \int_{-1}^{1} f(x) \, g(x) \, dx .$$

Claim This is a symmetric bilinear positive-definite form.
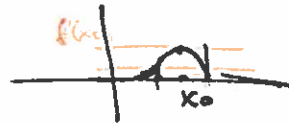
let's check linearity in the first argument:
$$(f_1 + f_2 , g) = \int_{-1}^{1} (f_1 + f_2)(x) \, g(x) \, dx$$

$$= \int_{-1}^{1} f_1(x) \, g(x) \, dx + \int_{-1}^{1} f_2(x) \, g(x) \, dx.$$

Similarly,
$$(\lambda f, g) = \lambda (f, g).$$

Positive-definite:
$$(f, f) = \int_{-1}^{1} f^2(x) \, dx \geq 0.$$

why $(f,f) \neq 0$ for $f \neq 0$:
since f is continuous, if $f \neq 0$
then exists $x_0 \in (-1,1)$ s.t. $f(x_0) \neq 0$
so $f^2(x_0) > 0$. Then $f^2 > 0$ on some
neighbourhood of x_0 since f is continuous
The integral over this neighbourhood will be
positive.

Additional
Comments ① In axiom (3) of the dot product,

there are two parts: $(x, x) \geq 0$ for all x

and : $(x, x) = 0 \Leftrightarrow x = 0$.

this part can be formulated over any field.
A bilinear form satisfying this is called non-degenerate

this part only makes sense over $\mathbb{R}$, because an arbitrary field has no notion of "positive"

Over an arbitrary field,

A symmetric bilinear non-degenerate form can be used for many purposes in the same way as the dot product.

② Over $\mathbb{C}$, it is often more convenient to use Hermitian forms:

$(,) : V \times V \longrightarrow \mathbb{C}$ is called a Hermitian form

if : i) $(x_1 + x_2, y) = (x_1, y) + (x_2, y)$

$(\lambda x, y) = \lambda (x, y)$

for $\lambda \in \mathbb{C}$

2) $(x, y_1 + y_2) = (x, y_1) + (x, y_2)$

$(x, \lambda y) = \overline{\lambda} (x, y)$ for $\lambda \in \mathbb{C}$

↑ complex conjugate of λ

3) $(x, y) = \overline{(y, x)}$ ← <u>Not</u> symmetric

4) $(x, x) \in \mathbb{R}$ and is non-negative

with $(x, x) = 0$ (⇔) $x = 0$.

Hermitian forms allow to make a theory for complex vector spaces that is in many ways analogous to the theory we will study for real Euclidean spaces.