Last time: Defined inner product $(,)$ on V ← a real vector space.

Makes it into Euclidean space. ↑ bilinear symmetric positive-def.

Example $(x,y) = x \cdot y$
$$= \underbrace{x_1y_1 + \cdots + x_ny_n}_{\text{dot product}}$$

Today:
- Cauchy-Schwartz inequality
- Triangle inequality
- orthogonal complements, orthogonalization.

Recall: we can use an inner product to define the notions of length: $\|x\| = \sqrt{(x,x)}$

and angle: 

$\cos\alpha = \dfrac{(x,y)}{\|x\|\,\|y\|}$

Need to prove: $|(x,y)| \le \|x\|\,\|y\|$

(otherwise, $\cos^{-1}(\cdot)$ will not be defined).

Last time: proved that for our usual dot product, the notion of angle def'd this way agrees with our usual angle (law of cosines)

of course also the usual length: $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ is exactly the norm that comes from the dot product.

Theorem $|(x,y)| \le \|x\|\,\|y\|$ ← Cauchy-Schwartz inequality

Pf: Fix x, y. Consider $(x-\lambda y,\ x-\lambda y) \geq 0$.

The book says: ~~[illegible]~~ Plug in $\lambda = \dfrac{(x,y)}{\|y\|^2} \in \mathbb{R}$

Proof 1

$$(x,x) - 2\lambda(x,y) + \lambda^2(y,y) \quad \leftarrow \boxed{\text{a degree 2 polynomial in } \lambda}$$

$$= \|x\|^2 - \frac{2(x,y)}{\|y\|^2}\cdot(x,y) + \frac{(x,y)^2}{\|y\|^{\cancel{4}\,2}}\cdot\cancel{\|y\|^2}$$

$$= \|x\|^2 - \frac{(x,y)^2}{\|y\|^2} \geq 0 \quad \text{so} \quad \boxed{\|x\|^2\|y\|^2 \geq (x,y)^2.}$$

Proof 2 (did this in class)

(I like this proof better)

Generally, $(x,x) - 2\lambda(x,y) + \lambda^2(y,y)$

$$= \underbrace{\overset{c}{\|x\|^2} - 2\lambda\overset{b}{(x,y)} + \lambda^2\overset{a}{\|y\|^2}}_{\text{quadratic in } \lambda} \geq 0 \quad \text{for all } \lambda.$$

/ **Recall:** $ax^2+bx+c \geq 0$
then $b^2-4ac \leq 0$. /

Then $4(x,y)^2 - 4\|x\|^2\|y\|^2 \leq 0$.

So, $(x,y)^2 \leq \|x\|^2\|y\|^2$

Example Consider the space of continuous functions on $[-1,1]$ (real-valued) with its inner product

$$(f,g) = \int_{-1}^{1} f(x)g(x)\,dx$$

(or: complex-valued functions, and define

$$(f,g) = \int_{-1}^{1} f(x)\overline{g(x)}\,dx \quad)$$

Then $(f,f) = \int_{-1}^{1} |f(x)|^2\,dx \in \mathbb{R}$ $\quad$ $(\bar{z} = a-bi$, $z = a+bi)$ $\quad$ check: $z\cdot\bar{z} = |z|^2$

(we discussed at length why this is an inner product - see below for a careful check).

For this inner product, we <u>just proved</u> that

$$\left(\int_{-1}^{1} f(x)\, g(x)\, dx\right)^2 \leq \left(\int_{-1}^{1} f(x)\, dx\right)^2 \left(\int_{-1}^{1} g(x)\, dx\right)^2$$

- this is what Cauchy - Schwarz inequality says.

(carefully done from last time)

<u>Example</u> of dot product in an infinite-dim. vector space

- consider the space of all continuous functions $f: [-1,1] \to \mathbb{R}$

They form a real vector space. (infinite-dim'l)

Define $$(f, g) = \int_{-1}^{1} f(x)\, g(x)\, dx.$$

<u>Claim</u> This is a symmetric bilinear positive-definite form.

let's check linearity in the first argument:

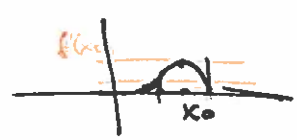
$$(f_1 + f_2, g) = \int_{-1}^{1} (f_1 + f_2)(x)\, g(x)\, dx$$

$$= \int_{-1}^{1} f_1(x)\, g(x)\, dx + \int_{-1}^{1} f_2(x)\, g(x)\, dx.$$

Similarly,

$$(\lambda f, g) = \lambda (f, g).$$

Positive-definite: $(f, f) = \int_{-1}^{1} f^2(x)\, dx \geq 0.$



why $(f,f) \neq 0$ for $f \neq 0$:

since f is continuous, if $f \neq 0$ then exists $x_0 \in [-1,1]$ s.t. $f(x_0) \neq 0$

so $f^2(x_0) > 0$. Then $f^2 > 0$ on some neighbourhood of x_0 since f is continuous. The integral over this neighbourhood will be positive.

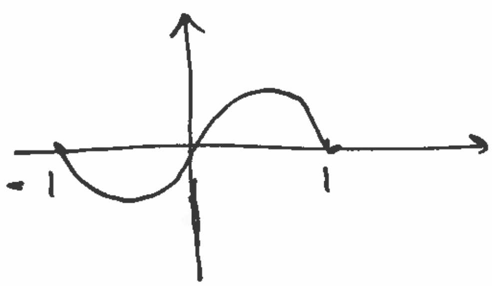
Thus we can define the "L^2-norm" of f:

$$\|f\|_2 = \sqrt{\int_{-1}^{1} f^2(x)\,dx}$$

The functions f, g are "orthogonal" if

$$(f,g) = 0 \quad \text{, i.e.} \quad \int_{-1}^{1} f(x)\,g(x)\,dx = 0 .$$

<u>Example</u>: $f(x) = \sin(\pi x)$, $g(x) = 1$ — orthogonal.

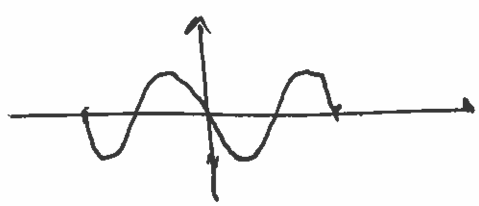


$\sin(\pi x)$, $\cos(\pi x)$ also orthogonal, but harder:

$$\int_{-1}^{1} \sin(\pi x)\cos(\pi x) =$$

use double angle formula

$$= \frac{1}{2}\int_{-1}^{1} \sin(2\pi x) = 0 .$$



Def: A basis $e_1, \dots, e_n$ in a Euclidean space V is called <u>orthonormal</u> if $(e_i, e_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

(all orthogonal, $\underbrace{\text{length } 1}_{\text{"normal"}}$)

"ortho" = orthogonal

Magic: If $e_1, \dots, e_n$ is an orthonormal basis then if $x = \lambda_1 e_1 + \dots + \lambda_n e_n$, you can find the λ's using the dot product:

$$\boxed{\lambda_i = (x, e_i)}$$