Today: More about linear trans. and matrices.

- Rank-nullity Theorem.

Announcements:

- Office hours
  Thursday 1-2 pm at MATH 217
  and on Zoom 6-7 pm
  (see Piazza for the link)

- Test on Friday
- Please bring your laptops on Friday
- No office hour on Friday
Proposition: Let $A: V \to W$ be a linear transformation.

Let $\{v_1, \ldots, v_n\}$ be a basis of $V$.

Then $A$ is completely determined by the vectors $A(v_1), \ldots, A(v_n) \in W$ and conversely, for any collection of vectors $u_1, \ldots, u_n \in W$, there exists exactly one linear transformation $B: V \to W$ s.t. $B(v_i) = u_i$ for $i = 1, \ldots, n$.

The rest is uniquely determined.

Proof: Recall that $\{v_1, \ldots, v_n\}$ is a basis of $V$.

For every vector $v \in V$, there exists unique collection $c_1, \ldots, c_n \in F$ s.t. $v = c_1 v_1 + \ldots + c_n v_n$.

So if $A(v_i) = u_i$, $A(v_n) = u_n$,

then $A(v) = c_1 u_1 + \ldots + c_n u_n$.

To do calculations in coordinates, we represent linear transformations with matrices $A \mapsto$ matrix of $A$. 
We recall how to associate a matrix with a linear transformation.

Let \( V \) have the basis \( \{ v_1, \ldots, v_n \} \).

Let \( W \) have the basis \( \{ w_1, \ldots, w_m \} \).

Let \( A: V \rightarrow W \) be a linear trans.

Then the matrix of \( A \) with respect to these bases is constructed as follows:

Write \( A(v_i) \) in coordinates with respect to the basis \( \{ w_1, \ldots, w_m \} \):

\[
A(v_i) = a_{1i} w_1 + a_{2i} w_2 + \cdots + a_{mi} w_m
\]

This keeps track of which basis vector \( v_i \) we are thinking about.

Now put \( (a_{1i}, \ldots, a_{mi}) \) as the \( i \)th column of the matrix:

\[
\begin{pmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{m1} & \cdots & a_{mn}
\end{pmatrix}
\]

\( f(v_i) \) \( f(v_m) \)
To check that this is, indeed, the matrix of the given linear operator, we just have to check that it does the same thing as the linear operator to the basis vectors (by the above proposition).

But this is so, basically, by definition:

\[
\begin{bmatrix}
    a_{11} & \cdots & a_{1n} \\
    \vdots & \ddots & \vdots \\
    a_{m1} & \cdots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
    \vdots \\
    1 \\
    \vdots \\
\end{bmatrix}
= \begin{bmatrix}
    a_{11}i_1 + \cdots + a_{1n}i_n \\
    \vdots \\
    a_{mi_1} + \cdots + a_{mn}i_n
\end{bmatrix}
\]

by def, the image of \( v_i \)

\( v_i \) in the coordinate system associated with the basis \( \{ v_1, \ldots, v_n \} \)
Example: Suppose we want to send the standard basis vectors in $\mathbb{R}^3$ to the vector $\mathbf{v} = (1,1,1)$. Geometrically, this will map $\mathbb{R}^3$ to the line spanned by $\mathbf{v}$.

Algebraically: Write the matrix:

$$f(e_1) = f(e_2) = f(e_3) = (1,1,1)$$

so we get the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$ 

Note that

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y + z \\ x + y + z \\ x + y + z \end{pmatrix}.$$ 

$$(x+y+z) \cdot \mathbf{v}$$

is a scalar in $\mathbb{R}$.

We note that the whole plane defined by the equation $x+y+z = 0$ maps to $0$. 

We have

$$(x+y+z) \cdot \mathbf{v} = 0$$

is a scalar in $\mathbb{R}$. 

We consider the line spanned by $\mathbf{v}$ in $\mathbb{R}^3$. 

The vector $\mathbf{v}$ is the projection of the vector $\mathbf{w}$ onto the line spanned by $\mathbf{v}$. 

The equation $x+y+z = 0$ defines a plane in $\mathbb{R}^3$. 

The plane intersects the line spanned by $\mathbf{v}$ at the point $\mathbf{0}$. 

We conclude that the plane $x+y+z = 0$ maps to the point $\mathbf{0}$.
Side notes: 1) the preimage of any point on the line $L$ is the whole plane parallel to the plane $x+y+z=0$ passing through the given point.

2) What would the matrix of that linear operator look like in a different basis?

- For an arbitrary basis, nothing special.
- But if we take a very special basis of the form $(v_1, v_2, u)$ where $(v_1, v_2)$ form a basis of the green plane $x+y+z=0$, then we'll get the following matrix:

$$
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

This is the vector $u$ in coordinates $\{v_1, v_2, u\}$:

$$
u = 0 \cdot v_1 + 0 \cdot v_2 + 1 \cdot u$$
Later in the course we will learn that for any linear operator there exists a basis in which its matrix looks especially nice.

Kernels, Images, Rank-Nullity:

**Def:** Let $A: V \rightarrow W$ be a linear transform. Then $\ker(A) = \{ v \in V \mid A(v) = 0 \}$ is the **kernel** of $A$.

$\text{Im}(A) = \{ A(v) \mid v \in V \}$ is the **image** of $A$.

The number $\text{rk}(A) = \text{dim}(\text{Im}(A))$ is called the **rank** of $A$.

$\ker(A)$ is a linear subspace of $V$.

$\text{Im}(A)$ is a linear subspace of $W$.

**Theorem** (dimension formula, rank-nullity theorem)

$$\text{rk}(A) + \dim(\ker(A)) = \dim(V)$$

the domain

**Example** In our example, $A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$\text{Im}(A) = L$ - the line spanned by $v$

$\ker(A) = \text{the plane } x+y+z=0$

$\dim(\ker(A)) + \text{rk}(A) = 2+1 = 3$. 