Today: • More about linear transf. and matrices.

• Rank-nullity Theorem.

Announcements: • Office hours Thursday 1–2 pm at MATH 217 and on zoom 6–7pm (see Piazza for the link)

• Test on Friday

• Please bring your laptops on Friday

• No office hour on Friday

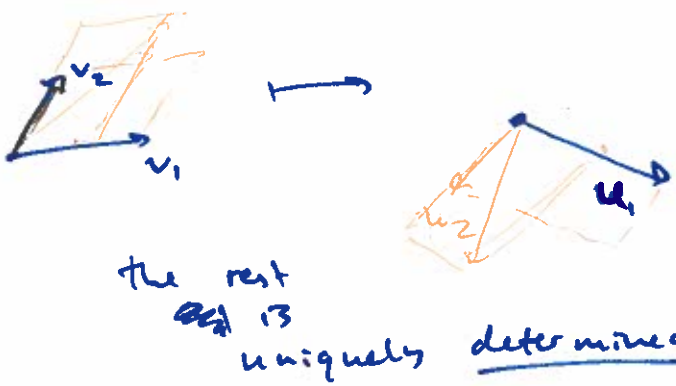
Proposition: Let $A : V \to W$ be a linear transformation.

~~Recall~~ Let $\{v_1, \ldots, v_n\}$ be a basis of V.

Then A is completely determined by the vectors $A(v_1), \ldots, A(v_n) \in W$

and conversely, for any collection of vectors $u_1, \ldots, u_n \in W$,

there exists exactly one linear transformation $B : V \to W$ s.t. $B(v_1) = u_1, \ldots, B(v_n) = u_n$.



Proof: Recall that $\{v_1, \ldots, v_n\}$ is a basis

(l.w.) $\Leftrightarrow$ for every vector $v \in V$, there exists unique collection $c_1, \ldots, c_n \in F$

s.t. $v = c_1 v_1 + \cdots + c_n v_n$

So if $A(v_1) = u_1, \ldots A(v_n) = u_n$,

then $A(v) = c_1 u_1 + \cdots + c_n u_n$.

To do calculations in coordinates, we represent linear transformations with matrices

$A \rightsquigarrow$ matrix of A.

We recall how to associate a matrix with a linear transformation:

Let V have the basis $\{v_1, \dots, v_n\}$

W have the basis $\{w_1, \dots, w_m\}$

Let $A : V \to W$ be a linear transf.

Then the matrix of A with respect to these bases is constructed as follows:

write $A(v_i)$ in coordinates with respect to the basis $\{w_1, \dots w_m\}$:

$$A(v_{\underline{i}}) = a_{1\underline{i}} w_1 + a_{2\underline{i}} w_2 + \cdots + a_{m\underline{i}} w$$

keeps track of which basis vector v_i we are thinking about.

Now put $(a_{1i}, \dots, a_{mi})$ as the i^{th} column of the matrix:

$$\begin{array}{ccc} \left[\begin{array}{c} a_{11} \\ \vdots \\ a_{m1} \end{array}\right. & \cdots & \left.\begin{array}{c} a_{1n} \\ \vdots \\ a_{mn} \end{array}\right] \\ f(v_1) & & f(v_n) \end{array}$$

To check that this is, indeed, the matrix of the given linear operator, we just have to check that it does the same thing as the linear operator to the basis vectors (by the above proposition)

But this is so, basically, by definition:

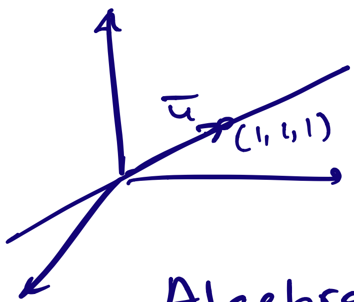
$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{bmatrix}$$

ith spot

this is the vector v_i in the coordinate system associated with the basis $\{v_1, \ldots, v_n\}$

by def, the image of v_i

Example Suppose we want to send the standard basis vectors in $\mathbb{R}^3$ to the vector $u = (1,1,1)$.



Geometrically, this will map $\mathbb{R}^3$ to the line spanned by $\bar{u}$.

Algebraically: write ~~the~~ the matrix:

$$f(e_1) = f(e_2) = f(e_3) = (1,1,1)$$

so we get the matrix

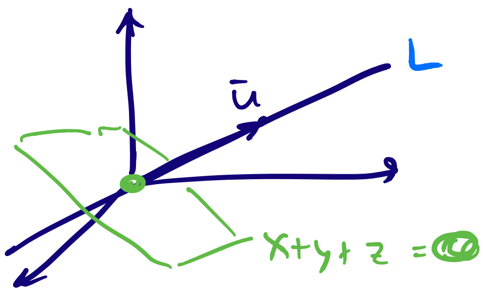
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Note that

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y+z \\ x+y+z \\ x+y+z \end{bmatrix}$$

$$= \underbrace{(x+y+z)}_{\text{scalar in } \mathbb{R}} \cdot \bar{u}$$

We note that the whole plane defined by the equation $x+y+z=0$ maps to 0.



<u>Side notes:</u> 1) the preimage of any point on the line L is the whole plane parallel to the plane $x+y+z=0$ passing through the given point

2) What would the matrix of that linear operator look like in a different basis?

- For an arbitrary basis, nothing special
- But if we take a very special basis of the form $\{v_1, v_2, u\}$ where $\{v_1, v_2\}$ form a basis of the green plane $x+y+z=0$, then we'll get the following matrix:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

image of v_1

image of v_2

This is the vector u in coordinates $\{v_1, v_2, u\}$:

$$u = 0 \cdot v_1 + 0 \cdot v_2 + 1 \cdot u$$

Later in the course we will learn that for any linear operator there exists a basis in which its matrix looks especially nice.

Kernels, Images, Rank-Nullity:

Def: Let $A: V \to W$ be a linear transf.
Then $\ker(A) = \{ v \in V \mid A(v) = 0 \}$ is the kernel of A

$\operatorname{Im}(A) = \{ A(v) \mid v \in V \}$ – is the image of A.

The number $rk(A) = \dim(\operatorname{Im}(A))$ is called the rank of A.

Exer $\ker(A)$ is a linear subspace of V
$\operatorname{Im}(A)$ is a linear subspace of W.

Theorem (dimension formula; rank-Nullity Thm)

$$rk(A) + \dim(\ker(A)) = \dim(V)$$

the domain

Example In our example, $A: \mathbb{R}^3 \to \mathbb{R}^3$
$\operatorname{Im}(A) = L$ – the line spanned by u
$\ker(A)$ = the plane $x+y+z=0$

$$\dim(\ker(A)) + rk(A) = 2+1 = 3.$$