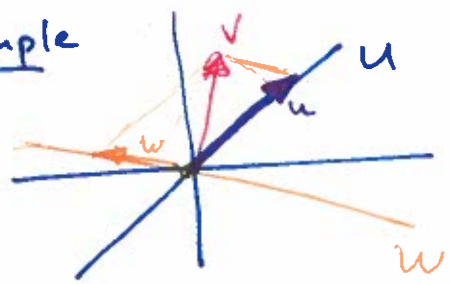
Today: rank of a (matrix) linear map.

Today: more on rank, and how to find it.

First, some examples, and more about linear maps.

Example



$\mathbb{R}^2$

Any two linear subspaces U, W of V such that

$$U \oplus W = V$$

(i.e., $U \cap W = \{0\}$ and $U + W = V$)

give rise to a projector:

$$P_U^W : V \to V$$ – projector onto U along W.

(exer: this works if and only if $V = U \oplus W$):

recall: (from HW) $V = U \oplus W \iff$ $\forall v \in V$ $\exists!$ $u \in U$, $w \in W$ s.t. $v = u + w$

exists unique

this says, every vector has well-defined components along U and W.

Define $\boxed{P_U^W(v) = u}$ ← forget the W-component.

Then by definition, $\mathrm{Ker}(P_U^W) = W$

$\mathrm{Im}(P_U^W) = U$.

This is a main example of a linear map with non-trivial kernel.

Dimension formula (Rank-nullity theorem) if $A: V \to W$ - linear map then

$$\boxed{\dim(\ker A) + \underbrace{\dim(\operatorname{Im} A)}_{rk(A)} = \dim(V)}$$

Pf: $\ker(A) \subset V$

Start with $v_1, \dots, v_k$ - basis of $\ker(A)$.

Let $v_{k+1}, \dots, v_n$ be vectors in V needed to complete it to a basis of V

(basis extension Thm).

Then look at $A(v_{k+1}), \dots, A(v_n)$ - images of $v_{k+1}, \dots, v_n$.

Exer: They are lin. indep. (in W) and span $\operatorname{Im}(A)$.

This completes the proof: this means,

$rk(A) = \dim(\operatorname{Im}(A)) = n - k$ $(k = \dim(\ker(A)))$.

More about matrices

Upshot: A vector space of dim n over F is identified with F^n by a choice of basis.

We have: $F^n = \{(a_1, \dots, a_n) \mid a_i \in F\}$.

standard basis $e_1, \dots, e_n$

$e_1 = (1, 0, \dots, 0), \dots$ $e_n = (0, \dots, 0, 1)$ notation only makes sense in F^n

Let V be a vector space over F, $\dim V = n$

Let $v_1, \dots, v_n$ be a basis of V.

Now can identify V with F^n:

Last time

vector spaces over a field F (finite-dim.)

- $A: V \to W$ - linear transformation

$V \cong F^n$, $W \cong F^m$

"isomorphic" (will define in a minute)

use the standard bases in both F^n, F^m — $\{e_1, \dots, e_n\}$, $\{e'_1, \dots, e'_m\}$

Then we can write A as a matrix! (a matrix for A)

The j^{th} column of A is the image of the basis vector e_j:

$$e_j \xmapsto{A} a_{j1} e'_1 + a_{j2} e'_2 + \dots + a_{jm} e'_m \quad \text{for } 1 \le j \le n$$

$$\begin{pmatrix} & \boxed{\begin{matrix} a_{j1} \\ \vdots \\ a_{jm} \end{matrix}} & \end{pmatrix}$$

j^{th} column

Def: If V, W are vector spaces over F, an isomorphism $A: V \to W$ is a linear transformation that has an inverse (as a function),

i.e. there exists $B: W \to V$ s.t.

"which means"

$$B \circ A = Id_V \qquad V \underset{B}{\overset{A}{\rightleftarrows}} W$$
$$A \circ B = Id_W$$

Exer: (h.w.) if such an inverse function B exists, it has to be a linear transformation!

If an isomorphism between V and W exists, they are called isomorphic vector spaces.

Will prove later today that <u>any</u> n-dimensional vector space is <u>isomorphic</u> to F^n.

Because of this, <u>all</u> linear transf. of finite-dim. vector spaces can be studied using matrices.

<u>How to compose linear transformations?</u>

$$\begin{tikzcd} F^n \arrow[r, "B"] \arrow[dr, "A \circ B"'] & F^m \arrow[d, "A"] \\ & F^r \end{tikzcd}$$

composition of lin. transf. is a lin. transf.

What is the matrix of the composition?

<u>Answer</u>: <u>matrix multiplication</u> (tricky to define).

<u>Example</u>: a lin. transf : $F^n \to F$ (no composition yet!)

$(\underbrace{a_{11}}_{\substack{\uparrow \\ \text{image of } e_1}} \dots a_{1n}) \leftarrow$ <u>one</u> row

When we apply it to a vector in F^n, we can write it as:

$$(a_{11} \dots a_{1n}) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \underset{\text{def}}{=} \underbrace{a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n}_{F}$$

the simplest matrix product

$\uparrow$ column vector in F^n (we always write vectors as columns!)

This does give a map from F^n to F.

Check it agrees with A: only need to check it does the right thing to the basis vectors.

Example: $\mathbb{R}^2 \xrightarrow{\begin{pmatrix}0&1\\1&0\end{pmatrix}} \mathbb{R}^2 \xrightarrow{A} \mathbb{R}^2$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \underset{\text{"}B}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \quad \cancel{\text{sort}} \;.$$

$$\underset{A}{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} \cdot \overset{\text{"}B}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{pmatrix}$$

"multiplying by this matrix has permuted the columns of A!

"permutation matrix"

In the other order: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$= \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix} \leftarrow \text{permuted the rows of } A!$$

Note: $AB \neq BA$!

Let us prove that it does correspond to the composition of linear transformations.

$$\underset{\{e_j\}}{F^n} \xrightarrow{B} \underset{\{e_j'\}}{F^m} \xrightarrow{A} \underset{\{e_j''\}}{F^r}$$

Need to show, that $A \circ B$ agrees with the one that has matrix AB

on the basis vectors e_j

$$e_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} b_{1j} \\ \vdots \\ b_{mj} \end{pmatrix} = b_{1j} \cdot e_1' + \cdots + b_{mj} e_m'$$

j^{th} spot

decomposition in the basis of F^m

j^{th} column of the matrix B (in F^m)

Take $e_j = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$ ← j-th place

$$(a_{11} \; \cdots \; a_{1n}) \cdot \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix} = 0 \cdot a_{11} + \cdots + 1 \cdot a_{1j} + 0 \cdots + 0$$
$$= a_{1j} \leftarrow \text{image of } e_j$$

so this formal product gives the right answer for each basis vector.

Def of matrix product

$A : F^m \to F^r$, $B : F^n \to F^m$ – linear transformations. We want to compute the matrix for their composition $A \circ B$ with respect to the standard bases.

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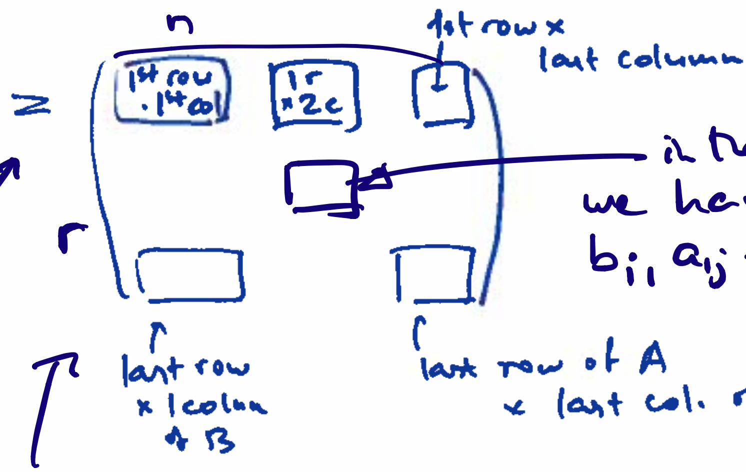
$$\begin{tikzcd} F^n \arrow[r, "A"] \arrow[dr] & F^m \arrow[d, "B"] \\ & F^r \end{tikzcd}$$

$$\underbrace{\begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \\ b_{r1} & \cdots & b_{rm} \end{pmatrix}}_{B} \cdot \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}}_{A}$$



in the ij-th place, we have $b_{i1} a_{1j} + b_{i2} a_{2j} + \cdots + b_{im} a_{mj}$

def of the matrix product AB

$\boxed{r \times n \text{ - matrix}}$

Optional: checking that the matrix product really corresponds to the composition.

For the basis vectors:

$$A \circ B(e_j) = A(b_{1j} e_1' + \cdots + b_{mj} e_m')$$

$$\underset{A \text{ is linear}}{=} b_{1j} A(e_1') + \cdots + b_{mj} A(e_m')$$

$$= b_{1j} \underbrace{\begin{pmatrix} a_{11} \\ \vdots \\ a_{r1} \end{pmatrix}}_{\substack{1^{st} \text{ column of } A \\ \text{(vector of length } r)}} + b_{2j} \begin{pmatrix} a_{12} \\ \vdots \\ a_{r2} \end{pmatrix} + \cdots + b_{mj} \underbrace{\begin{pmatrix} a_{1m} \\ \vdots \\ a_{rm} \end{pmatrix}}_{m\text{th column of } A.}$$

$$= \begin{pmatrix} b_{1j} a_{11} + b_{2j} a_{12} + \cdots + b_{mj} a_{1m} \\ b_{1j} a_{21} + \cdots + b_{mj} a_{2m} \\ \vdots \\ b_{1j} a_{r1} + \cdots + b_{mj} a_{rm} \end{pmatrix}$$

recognize this as the j^{th} column of AB:

the i^{th} entry is the product of the i^{th} row of A with the j^{th} column of B

Consider the product matrix AB, apply it to e_j:

$$\underbrace{\begin{pmatrix} c_{11} & \cdots & c_{1n} \\ & & \\ c_{r1} & \cdots & c_{rn} \end{pmatrix}}_{AB} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j^{th} \text{ spot} = \begin{pmatrix} \text{row } 1, j^{th} \text{ entry} \\ \vdots \\ j^{th} \text{ entry of row } r \end{pmatrix} = \boxed{j^{th} \text{ column of } AB.}$$

c_{ij}