So far, we have

- $A : V \to W$ - linear map $\rightsquigarrow$ get a matrix for A (depends on the bases)

  choose a basis in V — choose a basis in W

  (if $V = W$, $A : V \to V$, then has to be the same basis)

- A matrix $(m \times n)$ gives a linear map from F^n to F^m.

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \hline a_{m1} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} r_1 \cdot x \\ \vdots \\ r_m \cdot x \end{pmatrix}$$

write vectors as columns

$$r_i \cdot x = a_{i1} x_1 + \cdots + a_{in} x_n$$

- $\underset{n}{A : V} \to \underset{m}{W}$, $B : W \to \underset{r}{U}$ $\rightsquigarrow$ composition $B \circ A : V \to U$

  corresponds to the matrix product, BA.

  $r \times n$ - matrix.

- Kernels and images

  $$rk(A) = \dim(Im(A)) = \dim(\text{space spanned by columns of } A)$$

  $$\dim(\ker(A)) + rk(A) = \dim(V)$$

- Invertibility: $A : V \to W$ is invertible (A^{-1} exists) if and only if $\dim V = \dim W$, and

- A is bijective $\Leftrightarrow$ A is injective $\Leftrightarrow$ A is surjective $\Leftrightarrow \ker(A) = \{0\} \Leftrightarrow Im(A) = W$.

For a general linear operator,

In practice, we often care about $\underline{\ker(A)}$.

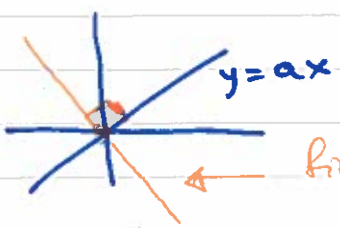
In fact, if you have dot product,

$\ker(A)$ = the space of all vectors in V perpendicular to all rows of A

(last class)

(This gives us geometric intuition in $\mathbb{R}^n$ but not so useful for computation).

$\mathbb{R}^2$



finding "orthogonal complement" is tricky.

<u>We need</u>: to solve equations. (7.1 in Jänisch)

We can encode systems of linear equations in this new language:

<u>Example</u>:
$$\begin{cases} x_1 - x_2 + x_3 = 1 \\ 2x_1 + x_2 - x_3 = 2 \\ 3x_1 + 2x_2 + x_3 = 0 \end{cases}$$

We can write it as:
$$\overbrace{\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}}^{\text{"}A\text{"}} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}_{b}$$

the rows are coefficients of the equations.

$\boxed{Ax = b}$ ← our system of linear equations

Possibilities for the solutions:

1. If $b = 0$, then
   The solutions form a linear subspace: the kernel of A.

2. Claim: If x_0 is one solution to $Ax = b$ then all solutions can be obtained as $x_0 + x$, where $x \in \ker(A)$.

Indeed, $A(x_0 + x) = Ax_0 + \underbrace{Ax}_{\substack{\shortparallel \\ 0}} = Ax_0 = b$. $\leftarrow x \in \ker(A)$

($\Leftarrow$)

($\Rightarrow$) Suppose y is a solution:

$$Ay = b$$

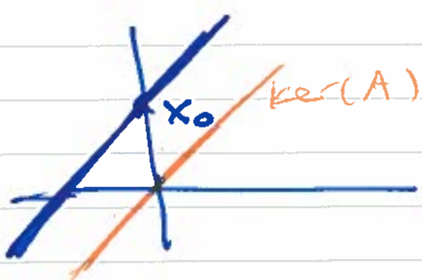
we also have $Ax_0 = b$.

Then $\underbrace{Ay - Ax_0}_{\substack{\shortparallel}} = 0$

$$A(y - x_0) = 0 \iff y - x_0 \in \ker(A)$$

$$\iff y = x_0 + \underbrace{(y - x_0)}_{\substack{\shortparallel \\ x \in \ker(A).}}$$

* The solutions of any system of linear equations form "a translated subspace" of V:



So, In $\mathbb{R}^3$, it is always:

- a single point
- a line
- a plane
- (or $\mathbb{R}^3$ if $A = 0$)

$\boxed{\text{Thus, we care about } \dim(\ker(A))}$

For any vector b on the right, the dimension of the "translated space" of solutions is the same! (= dim(ker(A)))

"number of parameters you need to describe all solutions.

- Finding the solutions in parametric form $\iff$ finding one solution x_0, and a basis of $\ker(A)$.

(parametric form: $x_0 + t_1 v_1 + \dots + t_k v_k$ where $\{v_i\}$ form a basis of $\ker(A)$).

- If we only care about dimension, enough to find $rk(A)$ (which is easier) and use that $\dim(\ker(A)) = \dim(V) - rk(A)$

How to do this

- Elementary row (and column) operations.

We will NOT do column ops.

3 operations:

1) Exchange two rows $\boxed{R_i \leftrightarrow R_j}$

2) multiply any row by $\neq 0$ scalar: $R_i \longrightarrow \lambda R_i$

3) take R_i, R_j, and add λR_j to R_i, leave R_j unchanged.

Example

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix} = A$$

illustrate (3): $\underline{R_2} - 2\underline{R_1}$ $\longrightarrow$ this becomes R

R_1 is untouched!

Get

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -3 & 2-2\cdot 1 = 0 \\ 3 & 2 & 1 & 0 \end{array}\right)$$

$-1-2\cdot 1 = -3$

$1 - 2\cdot(-1) = 3$

At the level of equations, we have:

the second equation now doesn't involve x_1.

This is elimination of variables!

Gaussian elimination

- exchange rows if needed to have $a_{11} \neq 0$.
  (maybe exchange columns if exchanging rows is not enough).
- Multiply R1 by $\frac{1}{a_{11}}$, to get the first entry to be 1.
- do operation (3) to rows to make the rest of the first column $= 0$. only do $R_i - \lambda \underline{R_1}$

In our example:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 1 & -1 & 2 \\ 3 & 2 & 1 & 0 \end{array}\right) \underset{R_2-2R_1}{\leadsto} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -3 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right) \underset{R_3-3R_1}{\leadsto} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & 5 & -2 & -3 \end{array}\right)$$

already 1

- Exchange things around to make $a_{22} \neq 0$, if possible.

- Divide R_2 by a_{22}, to get 1 there.

- Now use row operations to kill the second column (except a_{22})

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 5 & -2 & -3 \end{array}\right) \xrightarrow[\substack{R_1+R_2 \\ R_3-5R_2}]{\text{leave } R_2} \begin{array}{cccc} & \text{intended} & \text{luck} & \\ & \downarrow & \downarrow & \\ \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & -3 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -3 \end{array}\right) \underset{R_2+R_3}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

New system: $\begin{cases} x_1 = 1 \\ x_2 = -3 \\ x_3 = -3 \end{cases}$ – a solution!

<u>Did</u> 5.3, 7.1

W<u>ill do</u>: 5.4 next class.