Last time: we solved a system of linear equations

$$Ax = b \quad \text{with} \quad A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}$$

and $b = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ by doing Gaussian elimination.

It turned out that the system had unique solution.

Reminder: The structure of the set of solutions to the system of linear equations depends mostly on A, in the following sense:

- Solutions exist $\Leftrightarrow$ $b \in \mathrm{Im}(A)$

When the solutions exist,

- there is unique solution for some b $\Leftrightarrow$ there is at most one solution for any b

$\Leftrightarrow$ there is only one solution for $b = 0$ (and it is $x_1 = x_2 = \dots = x_n = 0$)

$\Leftrightarrow$ $\mathrm{Ker}(A) = \{0\}$.

If the matrix A is square these conditions are also equivalent to:

$\mathrm{Im}(A) = F^n$ $\Leftrightarrow$ for every b, there exists unique solution

(this is because of the rank-nullity Theorem).

The matrix A from last class was an example of a matrix satisfying these conditions.

Today: one more example (not of this kind)

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

Consider the system of equations $Ax = b$, where $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ (we think of it as a given vector of constants, but I do not want to specify any values for them).

We need to solve it for x (we keep the b_i as parameters)

We also want to know, for which $b = (b_1, b_2, b_3)$ does a solution exist?

We form an augmented system and do Gaussian elimination as last time:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{array}\right) \xrightarrow[\substack{R_2 - 4R_1 \\ R_3 - 7R_1}]{} \left(\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & -6 & -12 & b_3 - 7b_1 \end{array}\right)$$

$$\xrightarrow[-R_2/3]{} \left(\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & 2 & (-b_2 + 4b_1)/3 \\ 0 & -6 & -12 & b_3 - 7b_1 \end{array}\right)$$

$$\xrightarrow[R_3 + 6R_2]{} \left(\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & 2 & (-b_2 + 4b_1)/3 \\ 0 & 0 & \boxed{0} & \underbrace{b_3 - 7b_1 - 2b_2 + 8b_1}_{\text{“}b_3 + b_1 - 2b_2} \end{array}\right)$$

Note: this 0 happened "by coincidence"

What the last row tells us:

it gives the equation $0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = b_3 + b_1 - 2b_2$

This has a solution $\Leftrightarrow$ $\boxed{b_3 + b_1 - 2b_2 = 0}$

Now, we can do one more step to make this matrix more reduced: use the leading 1 in the second column to make a 0 above it:

$1 - 2R_2$:

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & b_1 - \frac{2}{3}(4b_1 - b_2) \\ 0 & 1 & 2 & -\frac{b_2}{3} + \frac{4}{3}b_1 \\ 0 & 0 & 0 & b_3 + b_1 - 2b_2 \end{array}\right)$$

Now we have a general solution:

it exists $\Leftrightarrow$ $b_3 + b_1 - 2b_2 = 0$,

and to find it, we set:

$$x_1 - x_3 = -\frac{5}{3}b_1 + \frac{2}{3}b_2$$

$$x_2 + 2x_3 = -\frac{1}{3}b_2 + \frac{4}{3}b_1$$

So we set: x_3 is a free parameter,

$$x_3 =: t$$

Then solve for x_2 and x_1:

$$x_2 = -\frac{1}{3}b_2 + \frac{4}{3}b_1 - 2t$$

$$x_1 = -\frac{5}{3}b_1 + \frac{2}{3}b_2 + t$$

We can write the solution in the parametric form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{5}{3} b_1 + \frac{2}{3} b_2 \\ \frac{4}{3} b_1 - \frac{1}{3} b_2 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

this is a basis vector for $\ker(A)$.

- Summary: to solve a system of linear eqn's, we get A to RREF:

$$\begin{pmatrix} 1 & 0 & * & * & 0 \\ 0 & 1 & * & * & 0 \\ \vdots & 0 & 0 & * & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

← rows of 0's at the end

pivot 1's

it might not be possible to have pivots in some columns.

For each of these columns, we declare one free parameter in the solution.

In this example, $x_3 = t$, $x_4 = s$ will be two free parameters, and $\ker(A)$ will be 2-dimensional.

We also discussed inverting a matrix, but see the next set of notes (Friday) for that — we filled in the details on Friday.