We recall:

RREF (Reduced Row Echelon Form) of a matrix looks like this:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots * & 0 & \cdots \\ 0 & 1 & 0 & * & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \hline 0 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix} \leftarrow \text{rows of 0's}$$

pivot 1's

If we started with a matrix for a system of linear equations: $Ax = b$, then:

- every column without a pivot gives a free variable
- If there are rows of 0's at the end, for the system $Ax=b$ to have a solution, b must satisfy the equation we get when doing row reductions for the augmented matrix:

$$\left(\begin{array}{c|c} A & \begin{matrix} b_1 \\ \vdots \\ b_n \end{matrix} \end{array}\right) \xrightarrow{\text{row ops}} \left(\begin{array}{c|c} \underbrace{\begin{matrix} 1 & \cdots & & \\ 0 & 0 & 1 & \cdot \\ 0 & 0 & 0 & 0 \end{matrix}}_{\text{RREF of } A} & \begin{matrix} \square \\ \\ \boxed{*} \end{matrix} \end{array}\right)$$

the combination of b_i's we get

So $b \in Im(A)$

$\iff Ax=b$ has a solution

$\iff * = 0$ holds for b

Conversely, for <u>every</u> b satisfying the equations we get from the rows of 0's, a solution exists:

indeed, look at RREF and solve them from the bottom up!

Suppose RREF looks like this:

$$\left(\begin{array}{cccc|c} 1 & 0 & * & 0 & \square \\ 0 & 1 & * & 0 & \square \\ 0 & 0 & 0 & 0 & \square \\ 0 & 0 & 0 & 0 & \square \end{array}\right)$$

some combinations of $b_1, \ldots, b_4$

We get: $\quad x_2 + * x_3 = \square$

declare x_3 a free variable, take it to the other side, get x_2.

Then: $\quad \underline{x_1} + \underline{* x_3} = \square$

Take x_3 to the other side, solve for x_1.

The shape of RREF guarantees that the solution always exists: as you go from the bottom up, you get a new variable to solve for, every line.

We obtain: Image of A is cut out *precisely* by the equations we get from the right-hand-side of the rows of 0's in RREF.

Then: $\operatorname{rk}(A) = \dim(V) - \#$ rows of 0's

(for a square matrix A)

(because non-zero rows are lin. indep.)

This gives another explanation of why row rank = column rank.

Note: elementary row operations correspond to multiplying the matrix A on the left by specific invertible matrices:

illustration for 2×2-matrices:

1) $R_1 \leftrightarrow R_2$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$

2) λR_1: $\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ c & d \end{pmatrix}$

3) $R_1 + \lambda R_2$: $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+\lambda c & b+\lambda d \\ c & d \end{pmatrix}$

Therefore, these operations do not change the column rank!

This example was <u>not</u> worked out in class, but it could be helpful anyway:

Example $A = \begin{pmatrix} 0 & 2 & 2 & 5 \\ 1 & -1 & 0 & 3 \\ 2 & 0 & 2 & 4 \end{pmatrix}$

- Find rk A, a basis for ker(A)

Solution: do Gaussian elim., get A into (reduced) row echelon form.

Row operations.

$$A \underset{R1 \leftrightarrow R2}{\rightsquigarrow} \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 2 & 2 & 5 \\ 2 & 0 & 2 & 4 \end{pmatrix} \underset{R_3 - 2R_1}{\rightsquigarrow} \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 2 & 2 & 5 \\ 0 & 2 & 2 & -2 \end{pmatrix}$$

$$\underset{\frac{1}{2}R_2}{\rightsquigarrow} \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 1 & 5/2 \\ 0 & 2 & 2 & -2 \end{pmatrix} \underset{R_3 - 2R_2}{\rightsquigarrow} \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 1 & 5/2 \\ 0 & 0 & 0 & -7 \end{pmatrix}$$

pivots

$$\underset{\frac{1}{-7}R_3}{\rightsquigarrow} \underbrace{\begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 1 & 5/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{row echelon form}} \underset{R_1 + R_2}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & 1 & 11/2 \\ 0 & 1 & 1 & 5/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

pivot 1's

now, go back kill all elements <u>above</u> the pivots

nothing to do

row echelon form (enough for finding rank: count the pivots)

$$\underset{\substack{R_1 - \frac{11}{2}R_3 \\ R_2 - 5/2 R_3}}{\rightsquigarrow} \underbrace{\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{reduced row echelon form}}$$

cannot improve it further w/o column operations.

we <u>do not</u> do column operations!

Analysis of the RREF: $\mathrm{rk}(A) = 3$ (3 pivots).

$A: \mathbb{R}^4 \to \mathbb{R}^3$, then $\mathrm{rk}(A) = 3$, $\dim(\ker(A)) = 1$.

Recall: $\dim(\ker(A)) + \mathrm{rk}(A) = \dim V = 4$

How to find the basis vector for $\ker(A)$?

After row reductions, we got:

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \\ x_4 = 0 \end{cases} \leftarrow \text{our reduced matrix } B \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3^rd column did not have a pivot

conditions for $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \ker(A)$

Get: (*) $\begin{cases} x_1 = -x_3 \\ x_2 = -x_3 \\ x_4 = 0 \end{cases}$ ← free variable (any value of x_3 gives you a solution)

columns w/o pivots give you free variables

$$\boxed{\#\text{ free variables} = \dim(\ker(A))}$$

So our kernel has basis $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$ (set $x_3 = 1$) plug into (*)

Any ~~element~~ element of $\ker(A)$ is of the form $\underbrace{x_3}_{\text{rename it "t"}} \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$

A Note about invertible matrices

When does a matrix A have an inverse?

1) it has to be square ($n \times n$)
(corresp. to a linear operator : $V \to W$ of the same dimension)

(very often, think of $A: V \to V$
$A: \mathbb{F}^n \to \mathbb{F}^n$)

2) And it has to be an isomorphism of vector spaces

Given $\dim V = \dim W$

$\iff \dim(\ker(A)) = 0$: $\ker(A) = \{0\}$ (injective)

$\iff rk(A) = n$ (i.e., A is surjective)

Equivalently, row reduced echelon form of A is the Identity!

(when we do row reductions, need to get a pivot in every column, otherwise it means $rk(A) < n$)

Example: (of a non-invertible matrix):

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

already reduced

another example: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 6 & 8 & 10 & 12 \\ * & * & * & * \end{pmatrix}$

rank ≤ 3 depending on the $*$

Inverting square matrices

Recall from last class

Ex: $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ – system of equations from last time.

$$Ax = b$$

we did: make an augmented matrix

$$(A \mid b) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 1 & -1 & 2 \\ 3 & 2 & 1 & 0 \end{array}\right)$$

b

$\underset{\text{elem. row operations}}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & & 0 & 1 \\ & 1 & & -3 \\ 0 & & 1 & -3 \end{array}\right) \rightsquigarrow \boxed{\begin{matrix} x_1 = 1 \\ x_2 = -3 \\ x_3 = -3 \end{matrix}}$

solution.
(unique!)

Note: in matrix form, we write it:
(in this case, $A: \mathbb{R}^3 \to \mathbb{R}^3$ is an isomorphism):

$$Ax = b$$

$$\underbrace{A^{-1} \cdot A}_{= Id} \cdot x = A^{-1} b$$

$$\boxed{x = A^{-1} b}$$

if A^{-1} exists (!)

(if we had A^{-1}, it would be easy to find the solution for every b)

How to invert a matrix using elementary row operations (5.4 in Jänisch)

$$\underbrace{\left(\begin{matrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{matrix}\right.}_{\text{our } A}\left|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right)$$

augmented: corresponds to solving three systems of equations at once:

$x = A^{-1}e_1 \leftarrow Ax = e_1$

$x = A^{-1}e_2 \leftarrow Ax = e_2$

$x = A^{-1}e_3 \leftarrow Ax = e_3$

first column of A^{-1}

do row operations to get $\left(\begin{matrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{matrix}\right|\underbrace{\quad\quad}\left.\right)$ whatever we get here is A^{-1}.

(exer.)

Why this works: row operations $\Leftrightarrow$ manipulating equations

get $\underset{A^{-1}b}{\mathrm{Id} \cdot x} = \underset{\text{new RHS.}}{(\quad)}$

clever observation: if we take $b_1 = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$b_2 = e_2$ ⋮

$b_3 = e_3$ ⋮

This will find $A^{-1}e_1 \leftarrow$ 1st column of A^{-1}

$A^{-1}e_2$

$\vdots$

$$\left(A \,\middle|\, \begin{matrix}1\\0\\0\end{matrix}\right) \rightsquigarrow \left(\mathrm{Id} \,\middle|\, \begin{matrix}*\\ *\\ *\end{matrix}\right)$$

solution to $Ax = \begin{pmatrix}1\\0\\0\end{pmatrix}$

$=$ exactly $A^{-1}\begin{pmatrix}1\\0\\0\end{pmatrix}$

$=$ 1st column of A^{-1}.

How can we get stuck?

Example $A = \begin{pmatrix}1 & -1 & 0\\ 2 & 0 & 2\\ 0 & 2 & 2\end{pmatrix}$

Try inverting it:

already 1. (otherwise get 1 there)

$$\left(\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 2 & 0 & 2 & 0 & 1 & 0\\ 0 & 2 & 2 & 0 & 0 & 1\end{array}\right) \xrightarrow[R_2-2R_1]{} \left(\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 0 & 2 & 2 & -2 & 1 & 0\\ 0 & 2 & 2 & 0 & 0 & 1\end{array}\right)$$

need to make it 1

$$\xrightarrow[\frac{1}{2}R_2]{} \left(\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 0 & 1 & 1 & -1 & \frac{1}{2} & 0\\ 0 & 2 & 2 & 0 & 0 & 1\end{array}\right)$$

$$\xrightarrow[R_3-2R_2]{} \left(\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 0 & 1 & 1 & -1 & \frac{1}{2} & 0\\ 0 & 0 & \boxed{0} & 2 & -1 & 1\end{array}\right)$$

M

! $\leftarrow$ matrix is NOT invertible

What does this mean:

1) some equations will have no solution:

$Ax = b$ ← depending on b, might have no solution:

e.g. $Mx = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ ← to have solutions, need 0 here after reductions

our A after reductions!

2) if after reductions, got $Mx = \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix}$

then we have: $x_1 - x_2 = c_1$
$x_2 + x_3 = c_2$

$M = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ from last page

$\xrightarrow{R_1 + R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ ← make it 0.

$\begin{cases} x_1 + x_3 = c_1 + c_2 \\ x_2 + x_3 = c_2 \end{cases}$

cannot get rid of these!

We got a 1-parameter family of solutions

$x_1 = c_1 + c_2 - x_3$ ← determines the "y" in the kernel of A.
$x_2 = c_2 - x_3$ ← "free variable" (can rename it "t" or smth.)

$x_0 = \begin{pmatrix} c_1 + c_2 \\ c_2 \end{pmatrix}$

$v = x_3 \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix} \in \ker(A).$

All this says, is:

$A : \mathbb{R}^3 \to \mathbb{R}^3$

we discovered that $\dim(\mathrm{Im}(A)) = 2$

$\dim(\ker(A)) = 1$

$Mx = \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix}$

(2-dim image.)

get 1-parameter family of solutions when we have them

row operations change the image but not its dimension.

The kernel: Recall that any solution has the form $x = x_0 + y$, $y \in \ker A$.

$Ax = b$

x_0: some one solution: $Ax_0 = b$

Row operations help us find ~~the~~ a basis for $\ker(A)$: here it is $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$

One more comment about $\mathrm{rank}(A) = \dim(\mathrm{Im}(A))$

- we could do column operations to find a basis for the image of A. (if the columns are lin. indep. they already give a basis for $\mathrm{Im}(A)$)
- After we did row operations to get A into "echelon form": $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ & & 1 \\ 0 & \cdots & 0 \end{pmatrix}$

0s

$\mathrm{rk}(A) = \#$ number of nonzero rows.