Determinants

**Example** 2x2-determinant.

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \] - a 2x2-matrix

\[ \det A = ad - bc \] - a scalar

The determinant formula for a 2x2-matrix is:

\[ \det A = ad - bc \]

**det** is a map: square matrices over \( F \) \[ \mathbb{F} \]

Let us talk about properties of this map for 2x2-matrices.

1) If you scale any row by \( \lambda \), \( \det(A) \) gets multiplied by \( \lambda \):

\[ \begin{bmatrix} \lambda a & \lambda b \\ c & d \end{bmatrix} \]

\[ \det = \lambda \det A \]

2) If we fix \( c, d \), replace \( a, b \) with \( a = a_1 + a_2 \)

\[ \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c & d \end{bmatrix} \]

\[ \det = \det \left( a_1 + a_2 \right) d - \left( b_1 + b_2 \right) c \]

\[ = ad - bc + a_2 d - b_2 c \]

\[ = \det \begin{bmatrix} a_1 & b_1 \\ c & d \end{bmatrix} + \det \begin{bmatrix} a_2 & b_2 \\ c & d \end{bmatrix} \]

(1) + (2) mean: "linear in each row" from last time.

3) \( \det \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \) but also \( \det(A) = 0 \) \( \Rightarrow \) \( \text{rk}(A) \leq 2 \)

\( \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0 \) \( \Rightarrow \) \( ad - bc = 0 \) \( \Rightarrow \) \( ad = bc \)

suppose, say, \( c \neq 0 \), \( d \neq 0 \)

Then \( \frac{a}{c} = \frac{b}{d} \)

Then rows are proportional. \( \Rightarrow \) \( \text{rk}(A) = 1 \)
If \( c = 0 \) and \( d = 0 \), we have a row of 0's, so \( \text{rk}(A) \leq 1 \).

If \( c = 0 \) and \( d \neq 0 \), then \( ad = 0 \) so \( a = 0 \). Then we have \( \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} \) - a column of 0's, so \( \text{rk}(A) \leq 1 \).

\[ \leq : \text{a reverse this argument.} \]

So:

\[ \text{det}(A) \neq 0 \iff \text{rk}(A) = 2 \iff A^{-1} \text{ exists!} \]

4) \( \text{det}(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = 1 \).

Now we can axiomatize this: say that we want a map \( D : n \times n \text{-matrices} \to F \) over \( F \) satisfying:

(1) linear in each row (our (1) and (2))
(2) \( D(A) \neq 0 \iff \text{rk}(A) = n \)
(3) \( D(\text{Id}) = 1 \).

Together, these properties guarantee that there is at most one such map \( D \).
In the lecture, we talked about replacing Axiom (1) with an easier-to-understand collection of requirements about how the map \( D \) should behave under the elementary row operations:

1) If we replace row \( R_i \) with a combination of \( R_i \) and \( R_j \):

\[
(R_i, R_j) \mapsto (R_i + cR_j, R_j)
\]

then \( D(A) \) stays the same.

2) If we multiply a row by a constant (scalar), \( D(A) \) multiplies by the same scalar:

\[
D \left( \begin{array}{c}
\begin{array}{ccc}
\alpha_{11} & \cdots & \alpha_{1n} \\
\vdots & \ddots & \vdots \\
\alpha_{m1} & \cdots & \alpha_{mn}
\end{array}
\end{array} \right) = \lambda D \left( \begin{array}{c}
\begin{array}{ccc}
\alpha_{11} & \cdots & \alpha_{1n} \\
\vdots & \ddots & \vdots \\
\alpha_{m1} & \cdots & \alpha_{mn}
\end{array}
\end{array} \right)
\]

3) If we swap two rows, \( D \) changes sign.

It is proved in the book that their set of axioms implies these.

The converse is also true; you are not required to prove it (but you can if you want).
We skip the proof that \( \dim(\text{space of maps satisfying } (1) \text{ and } (2)) = 6 \leq 1 \). It is in 6.1 (recommended reading).

We will build the det map. (which shows it exists!)

Start with \( 1 \times 1 \)-matrices:

1) \( (1 \times 1 \text{- matrices give linear maps } F \rightarrow F) \)
   
   a scalar: \( \begin{bmatrix} a \end{bmatrix} \in 1 \times 1 \text{- matrix}, \)
   
   \[ \det \begin{bmatrix} a \end{bmatrix} = a. \]
   
   \[ \det \begin{bmatrix} a \end{bmatrix} = 0 \iff a = 0. \]

2) \( 2 \times 2 \text{- matrices: we just did it.} \)
   
   \[ \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \cdot \det \begin{bmatrix} d \end{bmatrix} - b \cdot \det \begin{bmatrix} c \end{bmatrix} = ad - bc \]
   
   take an entry in the first row, and remove the column containing it.

3) \( 3 \times 3 \text{- matrices} \)
   
   \[ \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}
   + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \]

blocks alternate!
\[
= a_{11} (a_{22} a_{33} - a_{23} a_{32}) - a_{12} (a_{21} a_{33} - a_{23} a_{31}) \\
+ a_{13} (a_{21} a_{32} - a_{22} a_{31})
\]

Check that it agrees with:

\[
= a_{11} a_{22} a_{33} + a_{21} a_{32} a_{13} + a_{12} a_{23} a_{31} \\
- a_{13} a_{22} a_{31} - a_{11} a_{23} a_{32} \\
- a_{21} a_{12} a_{33}
\]

Each term has exactly one entry from each row and each column!

(This leads to an alternate description of \( \text{det} \) map, signs are hard to describe.)

\[ \text{or - ?} \]

See below

For \( n \times n \):

\[
\text{det} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11} \text{det } A_{11} \\
\text{minor of } a_{11}:
\text{obtained by throwing away row and column containing } a_{11}
\]

\[-a_{12} A_{12} + a_{13} A_{13} + \cdots + a_{1n} A_{1n} \leftrightarrow \text{ cofactor expansion in the first row.} \]
Key point:

- It doesn't change with elem. row/col ops of the form $R_i \rightarrow R_i + \lambda R_j$

- When you multiply a row or column by a scalar, det multiplies by that scalar.

- When you swap two rows or columns, det changes sign.

("easy" to derive from the recursive def'n.)

easy from the axioms.

So: we can do row ops to get a simpler matrix and get det that way.

Read: 6.1, 6.2