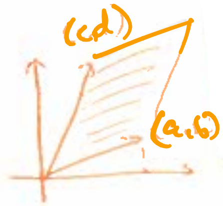
# Determinants




• "Example" $2\times 2$-determinant.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ - a $2\times 2$-matrix

formula for det $A$ for $2\times 2$-matrices

$\boxed{\det A \overset{def}{=} ad - bc}$ - a scalar (happens to equal area of a parallelogram)

"det" is a map : $\underset{\text{over } F}{\overset{n\times n}{\text{square matrices}}} \longrightarrow F$

Let us talk about properties of this map for $2\times 2$-matrices.

$\boxed{1)}$ If you scale any row by $\lambda$, det (A) gets multiplied by $\lambda$!

useful!

$$\begin{bmatrix} \lambda a & \lambda b \\ c & d \end{bmatrix} \xrightarrow{det} \lambda ad - \lambda bc = \lambda \det A.$$

2) if we fix $c, d$, replace $a, b$ with $a = a_1 + a_2$, $b = b_1 + b_2$

$$\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c & d \end{bmatrix} \xrightarrow{det} (a_1 + a_2)d - (b_1 + b_2)c$$
$$= a_1 d - b_1 c + a_2 d - b_2 c$$
$$= \det \begin{bmatrix} a_1 & b_1 \\ c & d \end{bmatrix} + \det \begin{bmatrix} a_2 & b_2 \\ c & d \end{bmatrix}$$

(1) + (2) mean: "linear in each row" from last time

3) $\det \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$. but also $\det(A) = 0 \iff rk(A) < 2$

$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0 \iff ad - bc = 0 \iff ad = bc$

suppose, say, $c \neq 0, d \neq 0$

Then $\frac{a}{c} = \frac{b}{d}$

Then rows are proportional. $\Rightarrow \underset{=1}{rk(A)}$

If $c=0$ and $d=0$, we have a row of 0's
so $rk(A) \leq 1$

if $c=0$ $d \neq 0$, then: $ad=0$ so $a=0$
then we have $\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}$ – a column of 0's
so $rk(A) \leq 1$.

$\Leftarrow$: reverse this argument.

so: $\boxed{\begin{aligned} \det(A) \neq 0 &\iff rk(A) = 2 \\ &\iff A^{-1} \text{ exists!} \end{aligned}}$

4) $\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$.

Now we can axiomatize this: we want / say that we

a map $D$: $n \times n$-matrices over $F$ $\to F$

satisfying: (1) linear in each row (our (1) and (2) from the example)

(2) $D(A) \neq 0 \iff rk(A) = n$

(3) $D(Id) = 1$.

this is from the book. will replace it with an equivalent axiom.

← requires proof

The maps satisfying these two properties form a 1-dim. vector space;

if $D_1$, $D_2$ are such maps, then $D_1 = c D_2$ for some $c \in F$.

"pins down" the const. $c$

Together, these properties guarantee that there is at most one such map $D$.

In the lecture, we talked about replacing Axiom (1) with an easier-to-understand collection of requirements about how the map D should behave under the elementary row operations:

1) If we replace row $R_i$ with a combination of $R_i$ and $R_j$ : $(R_i, R_j) \longmapsto (R_i + \lambda R_j, R_j)$

then $D(A)$ stays the same.

2) If we multiply a row by a constant (scalar), $D(A)$ multiplies by the same scalar:

$$D\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \boxed{\lambda a_{i1} \quad \cdots \quad \lambda a_{in}} \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \lambda D\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \cdots & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

3) If we swap two rows, D changes sign.

It is proved in the book that their set of axioms implies these.
The converse is also true; you are not required to prove it (but you can if you want).

we skip the proof that dim (space of maps satisfying (1) and (2) is $\underline{\underline{\leq 1}}$ )

it is in 6.1 (recommended reading)

We will <u>build</u> the det map. (which shows it exists!)

Start with 1×1-matrices :

1) (1×1- matrices give linear maps $: F \to F$ )

a <u>scalar</u> : $[a]$ ← 1×1 - matrix.

$a \in F$

$$\det([a]) = a.$$

$$\det([a]) = 0 \iff a = 0.$$

2) 2×2 - matrices: we just did it.

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \cdot \det[d] - b \det[c] = ad - bc$$

take an entry in the first row, and remove the column containing it

3) <u>3×3 -matrices</u>

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11} \det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$$- a_{12} \det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$+ a_{13} \det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

black block

$\boxed{\text{signs alternate!}}$

↑ blocks are "minors"

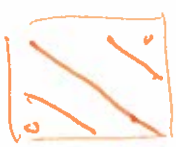
$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31})$$
$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Check that it agrees with:

another "wrapped diagonal"

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + \boxed{a_{21}a_{32}a_{13}} + a_{12}a_{23}a_{31}$$
$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32}$$
$$- \underbrace{a_{21}a_{12}a_{33}}$$

also "wrapped diagonal"



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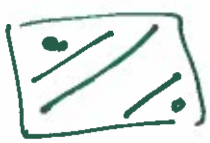

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each term has exactly one entry from each row and each column!

(this leads to an alternate description of 'det' map, signs are hard to describe.)

+ or − ?

see below

For $n \times n$:

$$\det \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & & \end{vmatrix} = a_{11} \det A_{11}$$

minor of $a_{11}$: obtained by throwing away row and column containing it

$$- a_{12} A_{12} + a_{13} A_{13} - \cdots \pm a_{1n} A_{1n}$$

cofactor expansion in the first row.

Key point:

- it doesn't change with elem. row/col ops of the form $R_i \to R_i + \lambda R_j$

- when you multiply a row or column by a scalar, det multiplies by that scalar.

- when you swap two rows or columns, det changes sign.

("easy" to derive from the recursive def'n.)
easy from the axioms.

So: we can do row ops to get a simpler matrix and get det that way.

Read: 6.1, 6.2