Recall determinants (6.1, 6.2)

- for a square matrix $A$, $\det(A)$ is a scalar.

- defined using axioms (as a map from matrices to $F$-scalars, it is linear in each row, $\det(A) = 0 \iff \text{rk}(A) < n$
  for an $n \times n$ matrix $A$
  $\det(\text{Id}) = 1$)

Today: want to get comfortable with computing det.

So far, we have:

1) $1 \times 1$ - matrix \( [a] \) $\det a.$

2) $2 \times 2$ - matrix: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$

\( \begin{vmatrix} a & b \\ c & d \end{vmatrix} \) means, $\det.$

Proof of the formula for area

- if you scale any side, area should multiply by the (abs. value of) the scale

- Area $= 0 \implies$ vectors are linearly dependent
  $\iff \text{rk} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \leq 1$
Area \((\square) = 1\).

Then: Area of \(\triangle (a,b,c,d)\) satisfies all the properties of \(\det\). Then it equals \(\det(A)\)!

3) For \(3 \times 3\) matrices:
   we have 3 ways of computing \(\det\):

1) Expansion by a row or column. \(\text{works for any size matrix}\)

\[
\begin{vmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 \\
\end{vmatrix} = +a_1 \begin{vmatrix}
  b_2 & b_3 \\
  c_2 & c_3 \\
\end{vmatrix} - a_2 \begin{vmatrix}
  b_1 & b_3 \\
  c_1 & c_3 \\
\end{vmatrix} + a_3 \begin{vmatrix}
  b_1 & b_2 \\
  c_1 & c_2 \\
\end{vmatrix}
\]

\(\text{signs}\)

2) Elementary row operations: \(\text{works for any size matrix}\)

- Interchange 2 rows \(\Rightarrow\) \(\det\) gets multiplied by \(-1\).
- Scale a row by a scalar \(\Rightarrow\) \(\det\) gets multiplied by that scalar
- \(R_i + cR_j\) (leave the \(j^{th}\) row unchanged, replace the \(i^{th}\) row with this sum)

These do not change \(\det\)!

- Once the matrix is upper-triangular,
  \[
  \begin{pmatrix}
  * & * & * \\
  0 & * & * \\
  0 & 0 & * \\
  \end{pmatrix}
  \]
  \(\det\) is easy: it is the product of the diagonal entries.
Example: \[
\begin{vmatrix}
1 & a & a^2 \\
1 & b & b^2 \\
1 & c & c^2 \\
\end{vmatrix}
\xrightarrow{R_2 - R_1}
\begin{vmatrix}
1 & a & a^2 \\
0 & b-a & b^2-a^2 \\
0 & c-a & c^2-a^2 \\
\end{vmatrix}
\]
(assume \(a, b, c\) are distinct, \(a, b, c \in \mathbb{F}\))

\[
\begin{align*}
R_3 & \xrightarrow{\frac{c-a}{b-a} \cdot R_2} \\
& \begin{vmatrix}
1 & a & a^2 \\
0 & b-a & b^2-a^2 \\
0 & 0 & c^2-a^2 - \frac{b^2-a^2}{b-a} \cdot (c-a) \\
\end{vmatrix}
\end{align*}
\]

upper-triangular

assuming \(b-a \neq 0\)

\[
= 1 \cdot (b-a) \cdot \left( c^2-a^2 - \frac{b^2-a^2}{b-a} \cdot (c-a) \right)
\]

\[
= (b-a) \cdot (c-a) \cdot (c-b)
\]

3) Just for 3x3: \[
\begin{vmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3 \\
\end{vmatrix}
= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2
- a_3b_2c_1
- b_1a_2c_3
- c_2b_3a_1
\]

can be generalized to any rank: combination of products one element from each row and each column with "tricky" \(\pm\) signs. Leibniz formula.

(handy for 3x3)
About the sign in Leibniz formula:

Each monomial \[ a_{j_1} a_{j_2} \ldots a_{j_n} \]

consists of a product of \( n \) elements of \( A \), exactly one from each row and each column.

We arrange them so that the row indices are in order, as above. Then the column indices give a permutation \( j_1, \ldots, j_n \) of the numbers \( \{1, \ldots, n\} \).

Example: in the 3x3 matrix

\[
\begin{pmatrix}
  a_{13} & a_{12} & a_{11} \\
  a_{23} & a_{22} & a_{21} \\
  a_{33} & a_{32} & a_{31}
\end{pmatrix}
\]

the monomial corresponding to the anti-diagonal \( a_{31} a_{22} a_{13} \) corresponds to the permutation \( (3, 2, 1) \).

Then we count the number of pairs in this permutation that are “out of order”:

\( (3, 2, 1) \) has \( (3, 1), (2, 1), (3, 2) \) of them.

The monomial comes with the sign \((-1)\) to the power of the number of out of order pairs.

In this example, it gives us \((-1)^3 = -1\).

\[ \det(A) \text{ is the sum of all such monomials with these signs.} \]
4) Geometric way:
\[
\begin{vmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{vmatrix} = \text{vol} \left( \mathbf{a} \times \mathbf{b} \times \mathbf{c} \right)
\]

Build a parallelepiped spanned by the rows of the matrix.

Aside
This tells us how to define volume in \(\mathbb{R}^n\):

\[
\text{Vol} \left( \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} \right) = \left| \text{matrix of components of the vectors spanning it} \right|
\]

(and this is what gives rise to change of variable in integral formula in multi-variable calculus)
Aside: How this is used in calculus

Change of variables formula in a multi-variable integral.

1) \( n = 1 \):

\[ \int h(y) \, dy = \int h(f(x)) \cdot f'(x) \, dx \]

\( y = f(x) \) "\( dy = f'(x) \, dx \)"

What happens here?

- \( f(x) \) scales every interval by \( c \), so if \( y = cx \), then \( dy = c \, dx \).

- Now, we approximate \( f \) by a linear function \( f'(x) \) at every point \( x \).

This is why if we set \( y = f(x) \), around every point \( x \), the interval of small length \( \Delta x \) becomes approximately an interval of length \( f'(x) \Delta x \).

- Now, in several variables:

Suppose you have a change of variables

in \( \mathbb{R}^2 \) :

\[
\begin{align*}
\theta &= f_1(x, y) \\
\sigma &= f_2(x, y)
\end{align*}
\]
To figure out what happens to area, you need to figure out approximately how big is the image at a small square around a given point \((x, y)\):

We approximate \(f\) by a linear operator around the point \((x, y)\). You know how to do it:

The matrix of this linear operator is the Jacobian matrix of \(f\):

\[
\begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{pmatrix}
\]
Then the image of the little square around \((x, y)\) is approximately the parallelogram you get by applying this matrix to your little square; that is, the determinant of the Jacobian matrix.

We set the change of variables formula:

\[
\iint h(s,t) \, ds \, dt = \iint g(x(t), y(t)) \left| \det \left( \begin{array}{cc} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{array} \right) \right| \, dx \, dy
\]

This holds for any number of variables (with an nxn-Jacobian matrix).

**Example** Polar coordinates:

\[
x = r \cos \theta \\
y = r \sin \theta
\]

Jacobian:

\[
\left( \begin{array}{cc} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{array} \right) = \left( \begin{array}{cc} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{array} \right)
\]
\[ \text{det (Jacobian)} = r \cos^2 \theta + r \sin^2 \theta = r \]

This is your \( r \) from the polar change.

If you don’t believe this works, compute the Jacobian for the spherical coords in \( \mathbb{R}^3 \):

\[
\begin{align*}
  x &= r \sin \phi \cos \theta \\
  y &= r \sin \phi \sin \theta \\
  z &= r \cos \phi.
\end{align*}
\]