Recall determinants (6.1, 6.2)

- for a square matrix A, $\det(A)$ is a scalar.

- defined using axioms (as a map from matrices to F ← scalars.

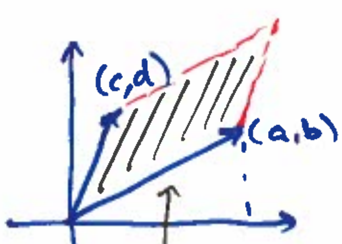
there exists exactly one map satisfying these properties →
$$\left\{\begin{array}{l} \text{it is linear in each row,} \\ \det(A) = 0 \iff rk(A) < n \\ \quad \text{for an } n\times n\text{-matrix } A \\ \det(Id) = 1 \end{array}\right)$$

Today: want to get comfortable with computing det.

So far, we have:

1) 1×1-matrix $[a] \xrightarrow{\det} a$.
2) 2×2-matrix: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \leftarrow \text{means, det.}$$

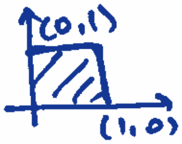


Area $= \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Proof of the formula for area

- if you scale any side, area should multiply by the (abs. value of) the scala

- Area $= 0 \iff$ vectors are linearly dependent $\iff rk \begin{bmatrix} a & b \\ c & d \end{bmatrix} \leq 1$

- 

Area () = 1.

- Then: Area of 

satisfies all the properties of det! Then it equals det(A)!

3) for 3x3-matrices:

we have 3 ways of computing det:

1) expansion by a row or column. ← works for any size matrix

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = +a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\boxed{\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}} \leftarrow \text{signs}$$

2) Elementary row operations: ← works for any size matrix

- interchange 2 rows ⟷ det gets multiplied by −1.
- scale a row by a scalar → det gets multiplied by that scalar
- $R_i + cR_j$ (leave the j^{th} row unchanged, replace the i^{th} row with this sum)

these do not change det!

- Once the matrix is upper-triangular, $\begin{pmatrix} * & & * \\ & * & \\ 0 & & \ddots \; * \end{pmatrix}$ det is easy: it is the product of the diagonal entries.

(Vandermonde determinant)

Example

(assume a,b,c are distinct) $a,b,c \in F$

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

kill this next

$$\xrightarrow[R_3-\frac{c-a}{b-a}R_2]{} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & c^2-a^2-\frac{b^2-a^2}{b-a}\cdot(c-a) \end{vmatrix}$$

upper-triangular

assuming $b-a \neq 0$

$$= 1\cdot(b-a)\cdot\underbrace{\Big(\underset{(c-a)(c+a)}{c^2-a^2} - \overset{b+a}{\frac{\cancel{b^2-a^2}}{\cancel{b-a}}}\cdot(c-a)\Big)}_{\substack{= (c-a)(c+\cancel{a}-(b+\cancel{a})) \\ = (c-a)(c-b)}}$$

$$= (b-a)(c-a)(c-b)$$

3) Just for 3×3:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + b_1c_2a_3 - a_3b_2c_1 - b_1a_2c_3 - c_2b_3a_1$$

can be generalized to any rank:

combination of products one element from each row and each column with "tricky" $\pm$ signs. ← Leibniz formula.

(handy for 3×3)

About the signs in Leibniz formula:

each monomial $\underline{a_{1j_1} a_{2j_2} \cdots a_{nj_n}}$

consists of a product of n elements of A, exactly one from each row and each column.
We arrange them so that the row indices are in order, as above. Then the column indices give a permutation $j_1, \ldots, j_n$ of the numbers $\{1, \ldots, n\}$. Example: in the 3×3-matrix the monomial corresponding to the anti-diagonal

$$\begin{pmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{pmatrix} \text{ is } a_{13}\,a_{22}\,a_{31}$$

corresponds to the permutation $(3\ 2\ 1)$

Then we count the number of pairs in this permutation that are "out of order":
in $(3\ 2\ 1)$ it is: $(3,1)$, $(2,1)$, $(3,2)$ - 3 of them.

The monomial comes with the sign
$$(-1)^{\#(\text{out of order pairs})}$$

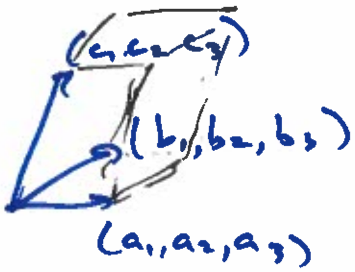
In this example, it gives us $(-1)^3 = -1$.

det (A) is the sum of all such monomials with these signs.

4) Geometric way:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \text{vol}\Big(\ \Big)$$

build a parallelepiped spanned by the rows of the matrix



(same proof as for area!)

<u>Aside</u>

This tells us how to define <u>volume</u> in $\mathbb{R}^n$.

Vol ( in $\mathbb{R}^n$) = | matrix of components of the vectors spanning it |

(and this is what gives rise to change of variable in integrals formula in multivariable calculus)

Aside   How this is used in calculus

Change of variables formula in a multi-variable integral.

1) $n=1$:

$$\int h(y)\,dy = \int h(f(x))\cdot f'(x)\,dx$$

$y = f(x)$ "$dy = f'(x)\,dx$"

What happens here?

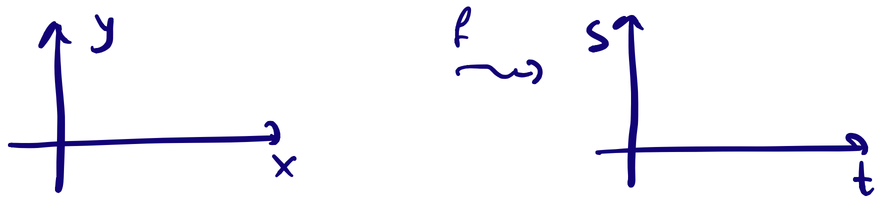
- a linear function $x \longmapsto cx$ scales every interval by c, so if $y = cx$, then $dy = c\,dx$.
- Now, we approximate f by a linear function of slope $f'(x)$ at every point x.

This is why if we set $y = f(x)$, around every point x, the interval of small length Δx becomes approximately an interval of length $f'(x)\,\Delta x$.

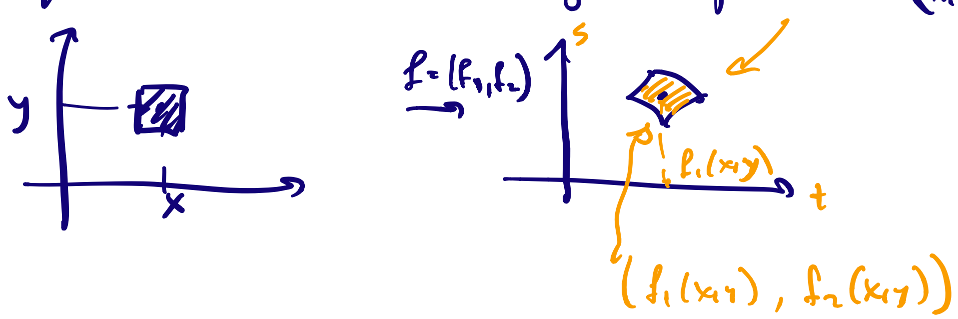
- Now, in several variables:

Suppose you have a change of variables in $\mathbb{R}^2$:

$$t = f_1(x,y)$$
$$s = f_2(x,y)$$



To figure out what happens to area, you need to figure out approximately how big is the image of a small square around a given point (x, y):



We <u>approximate</u> f by a <u>linear operator</u> around the point (x, y).
You know how to do it:
the <u>matrix</u> of this linear operator is the <u>Jacobian matrix</u> of f:

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$$

Then the image of the little square around (x,y) is approximately the parallelogram you get by applying this matrix to your little square: that is, the (abs. value of) determinant of the Jacobian matrix.

We get the change of variables formula:

$$\iint h(s,t)\,ds\,dt = \iint f(x,y)\left|\det\begin{pmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{pmatrix}\right| dx\,dy$$

$t = f_1(x,y)$
$s = f_2(x,y)$

This holds in any number of variables (with an $n \times n$-Jacobian matrix)

Example Polar coordinates:

$x = r\cos\theta$
$y = r\sin\theta$

Jacobian: $\begin{pmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{pmatrix} = \begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{pmatrix}$

$$\det(\text{Jacobian}) = r\cos^2\theta + r\sin^2\theta = \underline{\underline{r}}$$

this is your r
from the polar change

If you don't believe this works,
compute the Jacobian for the
spherical coords in $\mathbb{R}^3$:)

$$x = r\sin\varphi\cos\theta$$
$$y = r\sin\varphi\sin\theta$$
$$z = r\cos\varphi.$$