More properties of det

#0) **Exer:**

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{pmatrix} = 0 \leftarrow \text{Row } 3 = 2\times \text{Row } 1$$

recall: $\det \neq 0 \iff rk(A) = n \iff$ rows are lin. indep.
$\iff$ columns are lin. indep.

#1)

$$\begin{pmatrix} a_{11} & * & \cdots & * \\ 0 & a_{21} & \cdots & * \\ \vdots & 0 & \ddots & * \\ 0 & 0 & 0 & a_{nn} \end{pmatrix} \leftarrow \text{upper triangular}$$

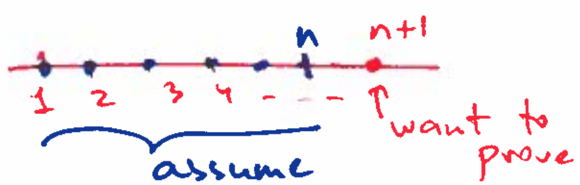
$\det(A) = a_{11} a_{21} \cdots a_{nn}$ – product of the diagonal entries

Pf: use induction

Aside: About induction, for statements of the form $P(n)$ ← depends on $n \in \mathbb{N}$.

(our case: $n \times n$ matrices).

method of proof: 1) prove it for some small n (usually, $n=1$)
(base case)



2) prove that $P(n) \Rightarrow P(n+1)$. "induction step"
Assume our statement is true for some $n \in \mathbb{N}$.
(or, for all $1, 2, 3, \ldots, n$)

- <u>Base case</u> $n=1$ $A=(a_{11})$ $\det(A) = a_{11}$, done

  [not needed, but as "sanity check", can also do: $n=2$:
  $A=\begin{bmatrix} a_{11} & b \\ 0 & a_{22} \end{bmatrix}$ $\det(A) = a_{11}a_{22} - 0 = a_{11}a_{22}$.
  Also true]

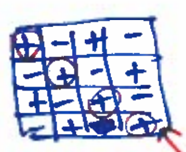
- <u>Induction step</u>: Assume we know the statement for $n\times n$-matrices. Let's prove it for $(n+1)\times(n+1)$-matrices.

Let $A = \begin{bmatrix} a_{11} & \cdots & & * \\ & \ddots & & \\ & & a_{nn} & \\ 0 & & & a_{n+1\,n+1} \end{bmatrix}$ be an $(n+1)\times(n+1)$-upper-triang. matrix.

use decomposition of det by the last row:

$$\det(A) = 0 + 0 - - + 0 + \underbrace{(-1)^{(n+1)+(n+1)}}_{\text{even!}} \det\underbrace{\begin{pmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}}_{= a_{11} \cdots a_{nn}} \cdot a_{n+1\,n+1}$$

$(-1)^{(n+1)+(n+1)} = +$

$\boxed{(-1)^{i+j}} \rightarrow$ 

note: all diagonal entries are +: $i+i = 2i$ - even!

$a_{11} \cdots a_{nn}$ by the induction assumption

$= a_{11} \cdots a_{nn} \, a_{(n+1)(n+1)}$

so we proved the formula for $n+1$.

3) (optional) "by the principle of mathematical induction" this completes the proof.

Another convenient kind of a matrix is "block upper-triangular":

$$\left(\begin{array}{c|c|c} A & * & * \\ \hline 0 & \boxed{B} & * \\ 0 & 0 & \boxed{C} \end{array}\right) \longleftarrow \text{anything}$$

square diagonal blocks

A special case of these is "block-diagonal":

$$\begin{pmatrix} \boxed{//} & 0 & 0 & 0 \\ 0 & \boxed{//} & 0 & 0 \\ 0 & 0 & \boxed{//} & 0 \\ 0 & 0 & 0 & \boxed{//} \end{pmatrix}$$

For these cases, det $=$ the product of the determinants of the diagonal blocks.

<u>Pf</u>: check the axioms.

or - think of volumes: for block-diagonal:

$$C = \begin{array}{c|cc} & v_1 \cdots v_k & v_{k+1} \cdots v_n \\ \hline v_1 \ \vdots & A & 0 \\ v_n & 0 & B \end{array}$$

this means, our basis is $v_1 \ldots v_k, v_{k+1} \ldots v_n$

and C takes $L(v_1, \ldots, v_k)$ to itself ← $C(v_1)$ has 0 entries for $v_{k+1} \ldots v_n$ so it lies in $L(v_1, \ldots, v_k)$

and takes $L(v_{k+1}, \ldots, v_n)$ to itself.

Another way to say it: we have $V = U \oplus W$

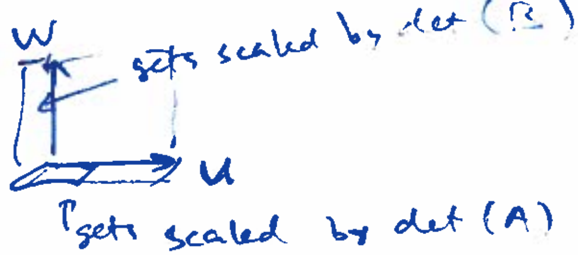
where $U = L(v_1, \ldots, v_k)$

$W = L(v_{k+1}, \ldots, v_n)$

and C takes U to U

W to W.

So what happens to volumes?



← volume scaled by det(A) det(B).