**Linear Transformations**

Def: \( V, W \) - vector spaces over \( F \).

\( f: V \to W \) is called a **linear transformation**

if: \( f(x+y) = f(x) + f(y) \) and \( f(\lambda \cdot x) = \lambda \cdot f(x) \) for \( \lambda \in F \).

(another word: **homomorphism** of vector spaces)

The set of all linear transformations from \( V \) to \( W \)

is denoted by \( \text{Hom}(V, W) \)

And \( \text{Hom}(V, W) \) is a vector space over \( F \)

(will put in homework)

**Examples**

1. \( \mathbb{R}^2 \to \mathbb{R}^2 \)

\[ f(x, y) = (2x + 3y, x - y) \]

\[ f: \mathbb{R}^2 \to \mathbb{R}^2 \]

\( (1,0) \to (2,1,1) \)

\( (0,1) \to (2,0+3,1, 0-1) \)

\( = (3, -1) \)

It is a lin. transf. (exer: check it!)

2. \( \mathbb{R}^2 \to \mathbb{R}^2 \)

\[ f(x, y) = (3x, 2y) \]

linear transformation

\( \mathbb{R}^2 \to \mathbb{R}^2 \)
b) \( f: \mathbb{R} \rightarrow \mathbb{R} \)
\( f(x) = x^2 \)
Not a linear transfor.
\( \text{e.g., because } (x+y)^2 \neq x^2+y^2 \)
\( (\lambda x)^2 \neq \lambda \cdot x^2 \)

same for \( f(x) = \frac{1}{x} \)
\( \Rightarrow f(x) = e^x \) \( f(x) = \sin(x) \)
NOT linear.

c) Trick question:
\( f: \mathbb{R} \rightarrow \mathbb{R} \) is a linear function: \( f(x) = ax + b \) where \( a, b \in \mathbb{R} \).

Then \( f \) is a linear transformation if and only if \( b = 0 \).

\( \text{PF: } \Rightarrow \) if a lin. transfor., \( \bullet \) want to prove \( b = 0 \)

\( \bullet \) apply it to \( \lambda \cdot x \):
Need to have:
\( a \cdot (\lambda x) + b = \lambda \cdot (ax + b) \) for \( \lambda \in \mathbb{R} \).

\( \lambda \cdot ax + b = \lambda \cdot ax + \lambda b \)
so \( (\lambda - 1)b = 0 \) for all \( \lambda \in \mathbb{R} \)
so \( b = 0 \).

\( \Leftarrow \) \( f(x) = a \cdot x \) → easy to check it is a lin. transfor.
• For more examples of linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$, see §4.6 in Jänisch.
(for the geometric point of view). We will also do a computer project about it.

Main points: a linear transformations have to take lines to lines (or points)
(and in higher dimensions, planes to lines or points, etc.)

• The set of fixed points of a linear transfor-
  mation must to be a linear subspace (exer).

A very important example: **Projector**:

![Diagram showing two lines and a point](image)

Let $L_1, L_2$ be lines in $\mathbb{R}^2$

passing through $(0,0)$.

The projector onto $L_1$ along $L_2$

can be defined geometrically:

for $x \in \mathbb{R}^2$, take a line

through $x$ parallel to $L_2$

and define $P(x)$ to be the intersection

point of that line with $L_1$.

(Thus, the whole line $L_2$ maps to $(0,0)$.)

The image of this map is $L_1$, and the preimage of any point on $L_1$ is a

line parallel to $L_2$. 


More examples: Let $V$ be the space of all functions $f: \mathbb{R} \to \mathbb{R}$.

Any point $a \in \mathbb{R}$ defines a linear transformation $A_a: V \to \mathbb{R}$ — "point evaluator": $f \mapsto f(a)$

**Def:** Let $V$ be a vector space over a field $F$. A linear transformation $f: V \to F$ is called a linear functional.

The example above is an example of a linear functional on the space of all functions.

**Example:** Let $a_1, \ldots, a_n \in \mathbb{R}$.

Consider the map $A: \mathbb{R}^n \to \mathbb{R}$ defined by:

$$ (x_1, \ldots, x_n) \mapsto a_1x_1 + \cdots + a_n x_n. $$

It is easy to remember the formula if we write it this way:

$$(a_1, \ldots, a_n) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = a_1 x_1 + \cdots + a_n x_n.$$

We will later prove that all linear functionals from $\mathbb{R}^n$ to $\mathbb{R}$ are of this form.

**Def:** $A: V \to W$ is called an isomorphism of vector spaces if it has an inverse map $B: W \to V$ s.t. $A \circ B = \text{Id}_W$, $B \circ A = \text{Id}_V$.

(Homework: prove that such a $B$ has to be linear.)
Example: we can use the above idea of how to write a linear functional on $\mathbb{R}^n$ to make a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$:

consider a matrix

$$
\begin{pmatrix}
 a_{11} & \cdots & a_{1m} \\
 \vdots & \ddots & \vdots \\
 a_{n1} & \cdots & a_{nm}
\end{pmatrix}
$$

and define, for $x = (x_1, \ldots, x_n)$

$$
A x := \begin{pmatrix}
 a_{11} & \cdots & a_{1m} \\
 \vdots & \ddots & \vdots \\
 a_{n1} & \cdots & a_{nm}
\end{pmatrix} \begin{pmatrix}
 x_1 \\
 \vdots \\
 x_n
\end{pmatrix}
$$

$$
\det \begin{pmatrix}
 a_{11} x_1 + a_{12} x_2 + \cdots + a_{1m} x_m \\
 \vdots \\
 a_{n1} x_1 + \cdots + a_{nm} x_m
\end{pmatrix}
$$

a vector with $m$ components, an element of $\mathbb{R}^m$.

It turns out that every linear transform from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be represented this way (will explain next class).