- Linear Transformations

Def: V, W - vector spaces over F.

$f: V \to W$ is called a linear transformation

if: $f(x+y) = f(x) + f(y)$ and

operations in V $\quad f(\lambda \cdot x) = \lambda \cdot f(x)$ for $\lambda \in F$.

operations in W

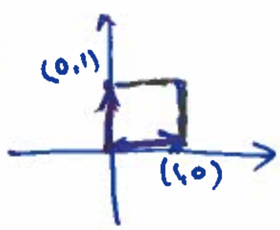
(another word: homomorphism of vector spaces.

The set of all linear transformations from V to W is denoted by $\mathrm{Hom}(V, W)$

And $\mathrm{Hom}(V, W)$ is a vector space over F

(will put in homework)

Examples

① 

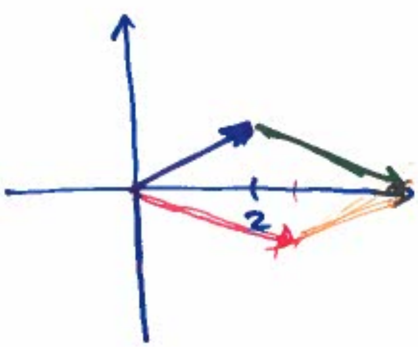
$f(x,y) = (2x + 3y,\ x - y)$

$f: \mathbb{R}^2 \to \mathbb{R}^2$

maps to $\quad$ tells us where the vector maps to

$(1,0) \mapsto (2 \cdot 1, 1)$

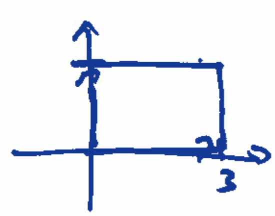
$(0,1) \mapsto (2 \cdot 0 + 3 \cdot 1,\ 0 - 1)$
$= (3, -1)$



It is a lin. transf.
(exer: check it!)

② 

$f(x,y) = (3x, 2y)$
linear transformation $\longrightarrow$



Easiest examples (1-dimensional)

0) a) $f: \mathbb{R} \to \mathbb{R}$

$f(x) = 3x$ – linear transf.

because: $3(x+y) = 3x+3y$
$3\cdot(\lambda x) = \lambda \cdot 3x$ in $\mathbb{R}$.

b) $f: \mathbb{R} \to \mathbb{R}$
$f(x) = x^2$

Not a linear transf.
e.g. because $(x+y)^2 \neq x^2+y^2$
$(\lambda \cdot x)^2 \neq \lambda \cdot x^2$

same for $f(x) = \frac{1}{x}$

$f(x) = e^x$ $\quad f(x) = \sin(x)$

NOT linear.

c) Trick question: $f: \mathbb{R} \to \mathbb{R}$ is a linear function: $f(x) = ax+b$ where $a, b \in \mathbb{R}$.

Then f is a linear transformation if and only if $b = 0$.

Pf: $(\Rightarrow)$ if a lin. transf., want to prove: $b = 0$

apply it to $\lambda \cdot x$:

Need to have: $a\cdot(\lambda x) + b = \lambda\cdot(ax+b)$ for $\lambda \in \mathbb{R}$.

$$\lambda a x + b = \lambda a x + \lambda b$$

so $(\lambda - 1) b = 0$ for all $\lambda \in \mathbb{R}$

so $b = 0$.

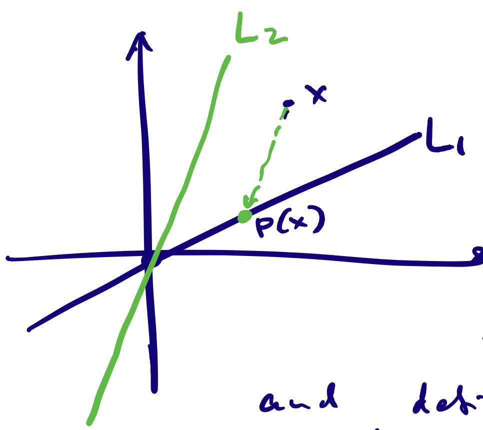
$(\Leftarrow)$ $f(x) = a\cdot x$ → easy to check it is a lin. transf.

- For more examples of linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$, see §4.6 in Jänich.
(for the geometric point of view). We will also do a computer project about it.

Main points: • linear transformations have to take lines to lines (or points)
(and in higher dimensions, planes to lines or points, etc.)

- The set of fixed points of a linear transf. has to be a linear subspace (exer).

- A very important example: Projector:



Let L_1, L_2 be lines in $\mathbb{R}^2$ passing through $(0,0)$.
The projector onto L_1 along L_2 can be defined geometrically: for $x \in \mathbb{R}^2$, take a line through x parallel to L_2 and define $P(x)$ to be the intersection point of that line with L_1.
(Thus the whole line L_2 maps to $(0,0)$)
The image of this map is L_1, and the preimage of any point on L_1 is a line parallel to L_2.

More examples Let V be the space of all functions $: \mathbb{R} \to \mathbb{R}$.

Any point $a \in \mathbb{R}$ defines a linear transf.

$$A_a : V \to \mathbb{R} \qquad \text{— "point evaluator".}$$
$$f \longmapsto f(a)$$

Def: Let V be a vector space over a field F. A linear transf. $f : V \to F$ is called a *linear functional*.

The example above is an example of a linear functional on the space of all functions.

Example: Let $a_1, \dots, a_n \in \mathbb{R}$. Consider the map $A : \mathbb{R}^n \to \mathbb{R}$ defined by: $(x_1, \dots, x_n) \longmapsto a_1x_1 + \dots + a_nx_n$.

It is easy to remember the formula if we write it this way:

$$(a_1 \dots a_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} := a_1x_1 + \dots + a_nx_n .$$

row of constants

column of variables

We will later prove that all linear functionals from $\mathbb{R}^n$ to $\mathbb{R}$ are of this form.

Def: $A : V \to W$ is called an *isomorphism of vector spaces* if it has an inverse map $B : W \to V$ s.t. $A \circ B = Id_W$, $B \circ A = Id_V$.

(Homework: prove that such a B has to be linear).

Example: we can use the above idea of how to write a linear functional on $\mathbb{R}^n$ to make a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$:

consider a matrix $\begin{pmatrix} a_{11} & - & - & a_{1n} \\ - & & -- & \\ a_{m1} & - & - & a_{mn} \end{pmatrix}$

and define, for $x = (x_1 \dots x_n)$

$$Ax := \begin{pmatrix} a_{11} & - & - & a_{1n} \\ \vdots & & & \\ a_{m1} & - & - & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\underline{\underline{\text{def}}} \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + - + a_{1n}x_n \\ - \quad - \\ a_{m1}x_1 + - \quad + a_{mn}x_n \end{pmatrix}$$

a vector with m components, an element of $\mathbb{R}^m$.

It turns out that every linear transf. from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be represented this way (will explain next class).