<u>Notation</u> 1) $\mathrm{Hom}_F(V,W)$ - the space of <u>all</u> linear transf. from V to W

(V and W are vector spaces over a field F)

2) $\mathrm{Mat}_{m\times n}(F)$ = the set of $m\times n$-matrices with entries in F.

Elements of this set look like this:

$$\begin{array}{cc} & n \text{ columns} \\ m \text{ rows} & \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ - & - & - \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \end{array} \qquad a_{ij}\in F \text{ for } 1\le i\le m,\ 1\le j\le n$$

We finished last time with the statement, in this language, that we have a map from $\mathrm{Mat}_{m\times n}(F)$ to $\mathrm{Hom}_F(F^n, F^m)$ defined by:

$$\underset{\substack{\cap \\ M_{m\times n}(F)}}{A} \longmapsto (x \longmapsto A\cdot x)$$

a linear transf from F^n to F^m

recall, $A\cdot x$ is defined by

$$\overset{A}{\overset{\shortparallel}{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ - & & - \\ a_{m1} & - & a_{mn} \end{pmatrix}}} \overset{\in F^n}{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} = \overset{\in F^m}{\begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ - \quad - \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix}}$$

a column of m numbers

Observation: The set $Hom_F(V, W)$ itself is a vector space over F with respect to the natural operations: $(A+B)(v) := A(v) + B(v)$

$\lambda \in F \quad (\lambda A)(v) = \lambda \cdot A(v)$

$A, B \in Hom_F(V,W)$ scalar mult. in W

(exer make sure you understand this statement. In lecture, we did this example:

$V = W =$ the space of smooth functions on $\mathbb{R}$

$D_1, D_2 \in Hom_{\mathbb{R}}(V, V)$

$D_1(f) = f'$, $\quad D_2(f) = f'' = D_1 \circ D_1$

Then $(D_1 + 3D_2)(f) \in Hom_{\mathbb{R}}(V, V)$

$(D_1 + 3D_2)(f) = f' + 3f''$.)

Example The homework problem about linear functionals can be reformulated as:

$$Hom_F(F^n, F) \cong F^n.$$

linear functionals, by def'n, are the elements of this space.

Main point of today

In a finite-dimensional vector space,

- choice of basis is the same as "choice of coordinates", which is the same as choosing an isomorphism from F^n to V.

Formally, let $\{v_1, \dots v_n\}$ be a basis of V. This choice of a basis determines the map

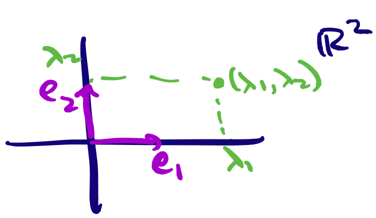
$$\Phi_{\{v_1,\dots v_n\}} : F^n \longrightarrow V$$

$$(\lambda_1, \dots \lambda_n) \longmapsto \lambda_1 v_1 + \dots + \lambda_n v_n$$

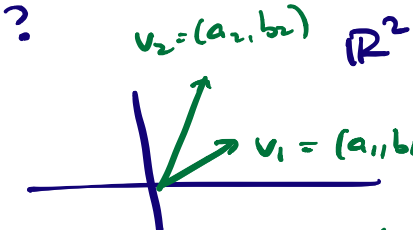
(notation from the book)

This map is an isomorphism by definition of a basis (exer: check this statement!)

Example What happens if we choose a different basis?



"old" basis: the standard basis $\{e_1, e_2\}$



"new" basis:
$v_1 = (a_1, b_1)$
$v_2 = (a_2, b_2)$

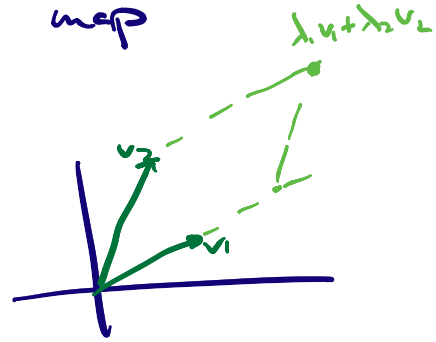
Note: (a_1, b_1) and (a_2, b_2) are the coordinates of the new basis vectors with respect to the old basis.

The first basis gives us the map

$$\Phi_{\{e_1,e_2\}} : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(\lambda_1, \lambda_2) \longmapsto \lambda_1 e_1 + \lambda_2 e_2 = (\lambda_1, \lambda_2)$$ – the identity

The second basis gives us the map

$$\Phi_{\{v_1,v_2\}} : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(\lambda_1, \lambda_2) \longmapsto \lambda_1 v_1 + \lambda_2 v_2$$



we can calculate the coordinates of this vector with respect to the "old" basis. (standard)

We get:

$$\lambda_1 v_1 + \lambda_2 v_2 = \lambda_1 (a_1, b_1) + \lambda_2 (a_2, b_2)$$
$$= \boxed{(\lambda_1 a_1 + \lambda_2 a_2,\ \lambda_1 b_1 + \lambda_2 b_2)}$$

It turns out that we can write this in a nice, easy-to-remember form, using matrices:

write all vectors as columns: $\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \in \mathbb{R}^2$

$v_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \leftarrow$ coords of v_1 in the 'old' basis

$v_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ - coords of v_2 in the 'old' basis.

Put v_1 and v_2 into a matrix as columns:

$$\left(\begin{array}{c|c} a_1 & a_2 \\ b_1 & b_2 \end{array}\right) =: C$$

v_1 v_2

Now apply this matrix to $\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$.

We get:

$$C\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \boxed{\begin{pmatrix} a_1\lambda_1 + a_2\lambda_2 \\ b_1\lambda_1 + b_2\lambda_2 \end{pmatrix}}$$

exactly the vector $\lambda_1 v_1 + \lambda_2 v_2$

The same works in any dimension.

We obtained:

if $\bar{v} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}^{old} \in V$ has coordinates $x_1, \ldots, x_n$ with respect to a basis $\{v_1^{old}, \ldots, v_n^{old}\}$

and $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}^{new}$ are its coordinates with respect to another basis $\{v_1^{new}, \dots v_n^{new}\}$

then we have:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}^{old} = C \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}^{new}$$

where C is the matrix formed from the coordinates of the new basis vectors in the old basis:

write $v_j^{new} = \begin{pmatrix} c_{1j} \\ \vdots \\ c_{nj} \end{pmatrix}$ ← coords in the old basis, i.e.,

$$v_j^{new} = c_{1j} v_1^{old} + c_{2j} v_2^{old} + \dots + c_{nj} v_n^{old}$$

and put these columns into a matrix:

coords of v_n^{new} in the old basis

$$C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix}$$

coords of v_1^{new} in the old basis

The matrix C is called "change of basis matrix".

Next: We want to associate a matrix with every linear transformation from V to W.

For that: • choose a basis $\{v_1 \ldots v_n\}$ of V

• choose a basis $\{w_1 \ldots w_m\}$ of W.

Given these bases, we can make a matrix for a given linear transformation.

Let $f : V \to W$ be a linear transf.

Consider the vectors $f(v_1), \ldots, f(v_n) \in W$.

Write them in coordinates with respect to the basis $\{w_1, \ldots, w_m\}$ of W:

$$f(v_j) = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix} \in W$$

this means, $f(v_j) = a_{1j} w_1 + \cdots + a_{mj} w_m$

Then put these columns into a matrix:

$$A := \left(\begin{array}{c|c|c|c} a_{11} & a_{12} & & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{array} \right)$$

$f(v_1)$ $f(v_2)$

Claim: the matrix A gives the linear transf. f:

$$A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$ means that

$$f(\underbrace{x_1 v_1 + \cdots + x_n v_n}_{\substack{\text{vector in } V \\ \text{with coordinates } \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\ \text{with respect to} \\ \text{the basis } \{v_1 \dots v_n\}}}) = \underbrace{y_1 w_1 + \cdots + y_m w_m}_{\substack{\text{vector in } W \\ \text{with coords} \\ \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \\ \text{in the basis} \\ \{w_1, \dots, w_m\}}}$$

Proof of the claim It is enough to prove it just for the basis vectors, because we know that a linear operator is determined by what it does to the basis vectors. But on the basis vectors $v_1, \dots v_n$ it is true by definition of the matrix A.