Extra credit assignment: review problems. Part 1.

Part 2 will be posted soon. The whole assignment is worth 5% and can be done instead of the computer project, or in addition to it for extra credit.

1. **Lines and planes in $\mathbb{R}^3$ and in higher dimension.** Remember from multivariable calculus that a line in $\mathbb{R}^3$ can be given by a parametric equation, or by "symmetric equation"

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

. Note that the "symmetric equation" is, in fact, *two* equations, and together they represent your line as an intersection of two planes, one vertical, one horizontal.

A plane in $\mathbb{R}^3$ (if it contains the origin) is a linear subspace of *co-dimension* 1 (so it is given by a single equation on x, y, z). Note that the way the equation for a plane is introduced in multivariable calculus uses the dot product: it is given via a normal vector (a, b, c) (then the equation for a plane containing 0 is

$$(x, y, z) \cdot (a, b, c) = 0,$$

which gives $ax + by + cz = 0$). However, in the end we do not need the dot product to think of a plane, it is just a linear subspace of $\mathbb{R}^3$ defined by one linear relation on its coordinates (in other words, it is a space of solutions of a single linear equation). Before proceeding, make sure you understand these statements.

In this problem, solve all the parts *without using cross product* (in $\mathbb{R}^3$, the cross product simplifies some of the calculations, but it does not generalize to higher-dimensional spaces; here the goal is to illustrate what happens in $\mathbb{R}^n$, so we do not want to be distracted by the cross product).

(a) Write down a general vector form of a parametric equation for a line in $\mathbb{R}^3$ that passes through the origin (namely, a 1-dimensional linear subspace of $\mathbb{R}^3$).

(b) Write down a general form of a parametric equation for a line in $\mathbb{R}^n$ that passes through the origin (namely, a 1-dimensional linear subspace of $\mathbb{R}^n$).

(c) Write down a general form of the "symmetric equation" for a line in $\mathbb{R}^4$. How many equations does it really consist of?

(d) If you take a "system" of linear equations that consists of a single equation $x + 3y + 2z = 0$ and "solve" it using the methods from our course, what do you get? (I claim that you should get a *parametric* equation for your plane, and it will use two parametrs). Interpret this solution in terms of a basis. Namely, find a basis for the plane $x + 3y + 2z = 0$.

(e) How many equations do you need to define a plane in $\mathbb{R}^4$? Why? Can you use rank-nullity theorem to give a rigorous proof?

(f) Find a basis for the plane W in $\mathbb{R}^4$ defined by the equations

$$x + 2y + 3z + w = 0$$
$$y - z - 5w = 0$$

(g) Interpret your result in the previous part as " The kernel of the linear map ... is the linear subspace W with basis ..." (Fill in the blanks in the above sentence).

(h) Write down a linear equation that defines the plane (call it W) with basis $\{(1, 1, 1), (2, 3, 1)\}$ in $\mathbb{R}^3$ (without using cross product!) I claim that I can interpret your result in the following language: "Find a *linear functional* $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathrm{Ker}(f) = W$." Write down this linear functional.

(i) Write down a system of linear equations in $\mathbb{R}^4$ that define the plane W with basis $\{(1, 1, 1, 1), (2, 3, 0, 1)\}$ and interpret your result as "$W = \mathrm{Ker}(A)$, where A is a linear map ....".

(j) Let $f_1 : \mathbb{R}^3 \to \mathbb{R}$ be the linear functional $f_1(x, y, z) = x + 2y + 3z$, and let $f_2 : \mathbb{R}^3 \to \mathbb{R}$ be the linear functional $f_2(x, y, z) = x - y - z$. Give a complete description of $\mathrm{Ker}(f_1) \cap \mathrm{Ker}(f_2)$ in as many ways as you can, as precisely as you can.

(k) How many linear equations do you need to define a plane in $\mathbb{R}^n$? Why?

(l) Decide whether the line with parametric equation $(x, y, z) = (t, 2t, 3t)$ is contained in the plane given by the equation $x + y - z = 0$.

Now I am going to ask this question in a different way: "Let $v = (1, 2, 3)$, and let w_1 and w_2 be a basis for the plane $x + y - z = 0$. Is the collection of vectors $\{v, w_1, w_2\}$ linearly independent? "

(m) Let W_1 be the subspace of $\mathbb{R}^5$ defined by the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 0$, and let W_2 be defined by the equations $x_1 - x_2 + x_3 = 0$ and $x_2 - 2x_3 + x_5 = 0$. Find the dimensions of W_1 and W_2, and find a basis for $W_1 \cap W_2$.

II. Linear maps and their matrices. The fundamental idea is that if you have a basis in V and a basis in W, you can define a linear map from V to W using a matrix. In the matrix, the columns are the coordinates of the images (in W) of the basis vectors of V. In doing this, you have complete freedom as to where your map sends the basis vectors.

1. Let P_n be the space of polynomials of degree not greater than n, and let $D : P_n \to P_n$ be the linear map given by $D(p) = p'$ (the derivative of p). Write down the matrix for D with respect to the basis $\{1, x, x^2, \ldots, x^n\}$. What is the size of this matrix? Find the kernel and image of D. Find the eigenvalues and eigenvectors of D in P_n. Prove that $D^n = 0$ (so D is a so-called *nilpotent* linear operator).

2. Curtis, Section 13, Problem 1 (p.107)

3. Curtis, Section 13, Problem 2 (p.107)

4. Explain whether there exists a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ such that $T(0,1,1,1) = (2,0)$, $T(1,2,1,1) = (1,2)$, $T(1,1,1,2) = (3,1)$, $T(2,1,0,1) = (2,3)$. (This is problem 11 in Curtis).

5. Let $C^\infty((0,1))$ be the space of all infinitely-differentiable functions on the interval $(0,1)$ (it is infinite-dimensional).

   (a) Prove that $e^{\lambda x}$ is an eigenvector of the differentiation operator $D : C^\infty((0,1)) \to C^\infty((0,1))$ defined by $D(f) = f'$. Conclude that the functions $e^{\lambda x}$ form a linearly independent set of vectors in $C^\infty((0,1))$.

   (b) Consider the linear subspace W of $C^\infty((0,1))$ spanned by $\{e^x, e^{2x}, e^{3x}\}$ (by the previous part, you know that they are linearly independent, so W is 3-dimensional). Let $A_1 : W \to \mathbb{R}^3$ be defined by $A_1(f) = (f(1), 3f'(2), f''(3))$. Find the matrix for the linear map A_1 with respect to the given basis of W and the standard basis of $\mathbb{R}^3$.