Recall last time:

- linear combinations
- **basis** of V: a set of vectors $v_1, \dots, v_n$ in V both **spanning**: $L(v_1, \dots, v_n) = V$ and **linearly independent**.

Our big goal:

- **Theorem**: Every basis has the same number of elements (when a finite basis exists).

Today:

- examples
- start proving this theorem (will need some lemmas).

Examples: of **linearly independent sets**:

1) **Main example**:

let F be an arbitrary field, and let $V = F^n = \underbrace{F \times \dots \times F}_{n \text{ times}}$.

Standard basis: (canonical)

elements of $\mathbb{F}^n$ as a set

basis $e_1 = (1, 0, 0 \ldots 0)$

$e_2 = (0, 1, 0 \ldots, 0)$

— —

$e_n = (0, 0 \ldots, 1)$

Why do they form a basis?

- check: 1) lin. indep.
  2) spanning. ← exer.

1) Suppose $\lambda_1 e_1 + \ldots + \lambda_n e_n = 0$. want to show: $\lambda_1 = \ldots = \lambda_n = 0$.

$$\lambda_1(1, 0 \ldots 0) + \ldots + \lambda_n(0, \ldots, 1)$$
$$= (\lambda_1, \lambda_2, \ldots, \lambda_n) \leftarrow \text{if this is } \bar{0}, \text{ then all } \lambda_i = 0.$$

Main point: ~~~~ $\mathbb{F}^n$ comes with a basis $\{e_1, \ldots, e_n\}$

point for the future: any n-dim. vector space over F is "isomorphic" to F^n. ("the same as")

aside: how big is a vector space of dimension n over $\mathbb{F}_p$?

~~ex~~: # V if dim $V = n$, over $\mathbb{F}_p$.:

e.g. dim $V = 2$. We have $\{v_1, v_2\}$ – basis of V over $\mathbb{F}_p$.

Every $v \in V$ has the form

$v = \lambda_1 v_1 + \lambda_2 v_2$ ← will soon prove λ_1, λ_2 are unique

p options for each

$\lambda_1, \lambda_2 \in \mathbb{F}_p$.

$$\# V = p^2$$

Can use induction to show that $\mathbb{F}_p^n$ has p^n elements.

Example 2: The space of all solutions to

$$f'' + f = 0$$

(subspace of all smooth real fns) ← infinite-dimensional

- $\sin x$, $\cos x$ satisfy it.
- Any solution is $c_1 \sin x + c_2 \cos x$ $\quad c_1, c_2 \in \mathbb{R}$

$\boxed{\text{2-dimensional}}$ (have not proved: there is no other solution lin. indep. from $\sin x$, $\cos x$)

Proofs:

easy lemmas:

- Call a set of linearly indep. vectors $\{v_1, \dots, v_n\}$ maximal if you cannot add any vector to it and keep it linearly independent.

Lemma: A maximal linearly indep. set is a basis.

Pf: We want to prove that maximal ⇒ spanning

Suppose it is not spanning.

Then exists $\vec{v}$ s.t. $v \neq \lambda_1 v_1 + \dots + \lambda_n v_n$ for any choice $\lambda_1, \dots, \lambda_n$

Then $\{v_1, \dots, v_n, v\}$ is still lin. indep.

So $\{v_1, \dots, v_n\}$ was not maximal.

More detail: suppose $\{v_1, \ldots, v_n, v\}$ is linearly dependent. Then we have

$$\lambda_0 v + \lambda_1 v_1 + \cdots + \lambda_n v_n = 0. \text{ for some } \lambda_0, \ldots, \lambda_n \in F.$$

We have two cases:

if $\lambda_0 = 0$, then $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$

which implies $\lambda_1 = \cdots = \lambda_n = 0$

by linear independence of $\{v_1, \ldots, v_n\}$.

Case 2: $\lambda_0 \neq 0$.

Then we can write

$$\lambda_0 \left(v + \frac{\lambda_1}{\lambda_0} v_1 + \cdots + \frac{\lambda_n}{\lambda_0} v_n \right) = 0$$

Then $v = -\left(\frac{\lambda_1}{\lambda_0} v_1 + \cdots + \frac{\lambda_n}{\lambda_0} v_n \right)$

Then $v \in L(v_1, \ldots, v_n)$, which contradicts our assumption on v.

Thus $\{v_1, \ldots, v_n, v\}$ is a linearly indep. set of vectors, and then $\{v_1, \ldots, v_n\}$ is not maximal.

We arrived at a contradiction, and thus proved that a maximal lin. indep. set must span all of V.