Today:
- Basis extension Theorem
- Basis exchange lemma.
- The notion of dimension

Basis exchange theorem:

Theorem: Suppose $\{v_1, \dots v_e\}$ is a lin. indep. set in V and $\{w_1, w_2 \dots w_s\} \subset V$ s.t.

$$L(v_1, \dots v_e, w_1, \dots, w_s) = V.$$

Then there exists a subset of $\{w_1, \dots, w_s\}$ s.t. together with $v_1, \dots v_e$ it forms a basis.

Proof: intuition: $\{v_1 \dots v_e\}$ - lin. indep.

either it is spanning, then we are done: take $\emptyset \subset \{w_1, \dots w_s\}$

if not spanning, we know that once all the w's are thrown in, it becomes a spanning set.

but maybe it has become lin. dependent.

So maybe we should not have put in all w's.

Then we try to put the w's in one-by-one:

find the w_i s.t. $w_i \notin L(v_1, \dots, v_e)$ (b/c all v's do not span but with the w's, they do.)

Put it in.

Because $w_i \notin L(v_1, \dots, v_e)$, the set $\{v_1, \dots, v_e, w_i\}$ is linearly independent.

Either $\{v_1, \dots v_e, w_i\}$ spans V, we are done

or it doesn't.

Then repeat the same argument.

In the book, this proof is done by induction on s: the number of w's.

(please read 3.4 - induction optional).

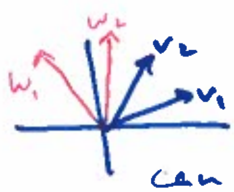
Basis exchange lemma

Suppose $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$ are both bases of a vector space V.

[should be the same as n but we have not proved it yet]

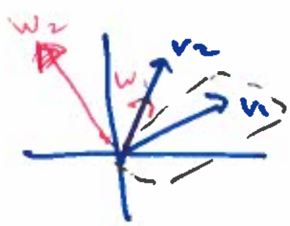
Then v_1 can be replaced with one of the w's, and $\{w_i, v_2, \dots, v_n\}$ will still be a basis.

Picture:



can make a basis out of one blue vector and one red vector.

less obvious case:



Proof: We are trying to exchange v_1 for one of the w's

Consider $L(v_2, \dots, v_n) \subset V$

Claim: there exists i, s.t. $w_i \notin L(v_2, \dots v_n)$

at least one of the w's is not in this subspace.

Pf of the claim: If all w_i lie in $L(v_2, \dots, v_n)$

then $L(w_1, \dots, w_m) \subseteq L(v_2, \dots v_n)$

But $\{w_1, \dots, w_m\}$ form a basis of the whole space V,
so this is impossible (unless $L(v_2, \dots v_n) = V$,
but this cannot happen because
$\{v_1, \dots, v_n\}$ was a basis, so
linearly indep.; and if
$L(v_2, \dots, v_n) = V$, then
$\{v_1, v_2 \dots v_n\}$ would have been
linearly dependent).

So let i be such an index that $w_i \notin L(v_2, \dots, v_n)$.

Then <u>claim</u>: $\{w_i, v_2, \dots v_n\}$ is a basis of V.

We know: $\exists\ \lambda_1, \dots, \lambda_n \in F$ s.t.

$$w_i = \lambda_1 v_1 + \dots + \lambda_n v_n$$

and $\lambda_1 \neq 0$ because otherwise w_i would be in $L(v_2, \dots, v_n)$

Then $$v_1 = \frac{1}{\lambda_1}\left(w_i - \lambda_2 v_2 - \dots - \lambda_n v_n\right)$$

Then $V = L(v_1, \dots, v_n) \subseteq L(w_i, v_2, v_3, \dots, v_n)$

Then $\{w_i, v_2, \dots, v_n\}$ is a spanning set for V,
and it is lin. indep. b/c $w_i \notin L(v_2, \dots, v_n)$
and $v_2 \dots v_n$ are independent.

So it is a new basis.

Proposition: 1) If V has a basis of n elements, $\boxed{\dim(V) = n}$ then every basis of V has n elements.

2) Any collection of vectors of V that has more than n elements is linearly dependent.

Pf: 2) Suppose $\{v_1, \dots, v_n\}$ is a basis
Suppose that $\{w_1, \dots, w_{n+1}\}$ is a lin. indep. collection of vectors

Then $\underbrace{\{w_1, \dots w_{n+1}, v_{s_1}, v_{s_2}, \dots v_{s_k}\}}$ is a basis for some $s_1, \dots, s_k$ by the basis extension Thm

put some of the v's together with w's. if needed.

Now I can exchange w_1 for one of the v's by the exchange lemma. Will get a new basis. Keep exchanging untill all the w's are gone. Will get a list of vectors where some of the v's are listed more than once!

This cannot be a linearly independent collection! We got a contradiction — there could not have been $n+1$ lin. indep. w's.

(2) ⇒ (1). Suppose we have two bases with different numbers of elements — say, one with n elements and one with m elements. Without loss of generality, we can assume that

$m < n$.

Then by (2), the collection of vectors with m elements would have to be linearly dependent, contradicting that it was a basis.