**Recall** A basis of a vector space $V$ is:

- a maximal linearly independent set of vectors,

which is equivalent to being

- a minimal spanning set.

We say that $V$ is finite-dimensional if it has a finite basis.

**Today**: 1) Example of an infinite-dimensional vector space

2) Direct sums, and the dimension formula.

① An important infinite-dim. space over $\mathbb{R}$:

Consider the space of all functions $f: \mathbb{R} \to \mathbb{R}$.

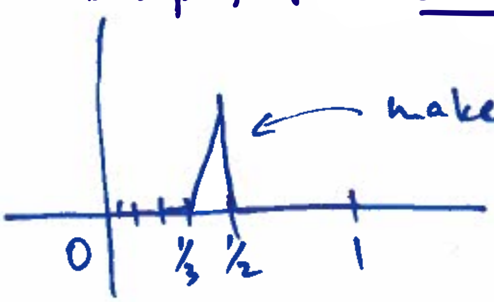
We can put various conditions:

- continuous, or
- smooth or
- integrable.

All these spaces are still infinite-dimensional.

To prove this, all we have to do is make an example of an infinite linearly independent system of functions in our space.

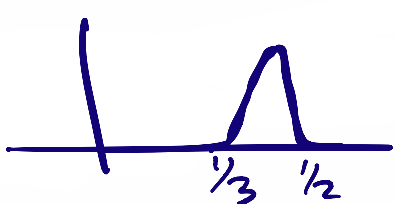
For example, for continuous functions,




make a function $\neq 0$ on $\left[\frac{1}{n+1}, \frac{1}{n}\right]$

$= 0$ elsewhere.

(You could also smoothen this to make it look like this:

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Why are such functions linearly independent?

First, what does it mean for an infinite set of vectors to be linearly independent:

we say a collection $W \subset V$ is linearly independant if $\forall\, w_1, \dots, w_n \in W$

$\lambda_1, \dots, \lambda_n \in F$, $\lambda_1 w_1 + \dots + \lambda_n w_n = 0$

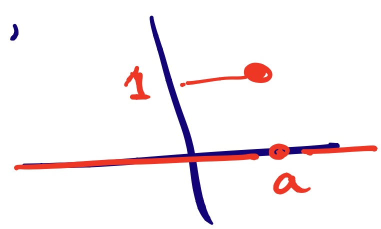
implies that $\lambda_1 = \dots = \lambda_n = 0$

(ie., no finite linear combination of vectors in $W$ is allowed to be $0$ unless all the coefficients are $0$).

The main example of a lin. indep. infinite set in the space of all functions $f: \mathbb{R} \to \mathbb{R}$: the delta-functions:

Define, for every $a \in \mathbb{R}$,

$$\delta_a(x) = \begin{cases} 1 & x = a \\ 0 & x \neq a \end{cases}$$




The collection $\{\delta_a(x) \mid a \in \mathbb{R}\}$ is lin. indep. (will check in a moment)

The above example is a "smoothening"

of this example.

Why are the delta-functions linearly independent:

suppose $\lambda_1 \delta_{a_1} + \text{---} + \lambda_n \delta_{a_n} = 0$

zero function

i.e. it equals 0 at <u>every</u> $x \in \mathbb{R}$

Then, in particular, at $x = a_i$, we have

$$\underbrace{\lambda_1 \delta_{a_1}(a_i) + \text{---} + \lambda_n \delta_n(a_i)}_{\parallel} = 0$$

$$0 + \text{---} + \underbrace{\lambda_i 1}_{\lambda_i \delta_{a_i}(a_i)} + \text{---} + 0 = 0$$

Thus, $\lambda_i = 0$ for all $i$.

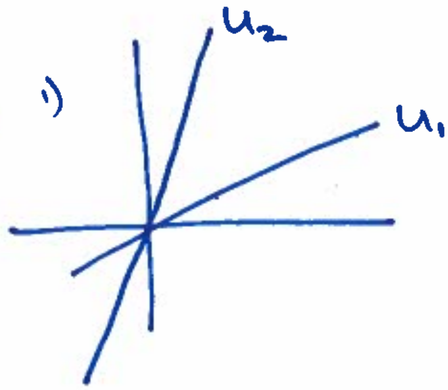
Next: sums and direct sums of 2 linear subspaces.

**Def:** Let $U_1, U_2$ - linear subspaces of $V$.

$$U_1 + U_2 = \{ x+y \mid x \in U_1, y \in U_2 \}.$$

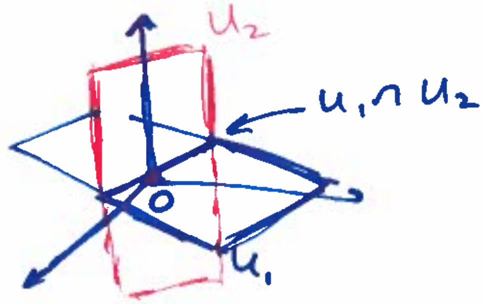
**Exer:** check that $U_1 + U_2$ is a linear subspace of $V$.

**Examples:** 1)




$U_1 + U_2 = \mathbb{R}^2$

2) in $\mathbb{R}^3$:




$U_1 + U_2 = \mathbb{R}^3$.

**Prop:** Let $U_1, U_2$ be linear subspaces of $V$.
assume $U_1, U_2$ are finite-dimensional.

check: $U_1 \cap U_2$ is a lin. sub

Then
$$\boxed{\dim(U_1 + U_2) + \dim(U_1 \cap U_2) = \dim(U_1) + \dim(U_2)}$$

**Check:** in our example in $\mathbb{R}^3$:

$\dim(U_1) = \dim(U_2) = 2$

$\dim(U_1 \cap U_2) = 1.$

$\dim(U_1 + U_2) = 3.$

**Question:** in $\mathbb{R}^4$, do two planes containing $0$ have to have a common line?

Yes: $\mathbb{R}^4 = L(e_1, \ldots, e_4)$. Take $U_1 = L(e_1, e_2)$
$U_2 = L(e_3, e_4)$

**Def:** If $U_1 \cap U_2 = \{0\}$ and $U_1 + U_2 = V$ then we say that $V$ is a "direct sum" of $U_1$ and $U_2$, write $V = U_1 \oplus U_2$

\oplus

In our $\mathbb{R}^4$ example, check:

$$\mathbb{R}^4 = L(e_1, e_2) \oplus L(e_3, e_4)$$

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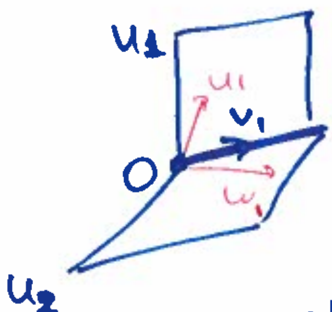
## Pf of the dimension formula

Consider $U_1 \cap U_2$; let $v_1, \ldots, v_s$ be its basis.




Use basis extension to extend it to $\{v_1, \ldots, v_s, u_1, \ldots, u_\ell\}$ – basis of $U_1$

$\{v_1 \ldots v_s, w_1 \ldots w_k\}$ – basis of $U_2$

Claim: $\{v_1, \ldots v_s, u_1, \ldots u_\ell, w_1 \ldots w_k\}$ is a basis of $U_1 + U_2$.

The formula follows:

$$\dim(U_1 + U_2) = s + \ell + k$$

$$\dim(U_1) + \dim(U_2) = (s + \ell) + (s + k).$$

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**Proof of the claim**: why this set spans $U_1 + U_2$: exercise.

## Why is this set linearly independent

Suppose we had some linear combination

$$\lambda_1 v_1 + \cdots + \lambda_s v_s + \mu_1 u_1 + \cdots + \mu_\ell u_\ell + \nu_1 w_1 + \cdots + \nu_k w_k = 0$$

(I am using different letters : $\lambda, \mu, \nu$ for scalars to make it easier to keep track of which vectors they come with).

We need to prove that all $\lambda$'s, $\mu$'s and $\nu$'s are 0.

We have:

this vector is in $U_1$

$$\lambda_1 v_1 + \cdots + \lambda_s v_s + \mu_1 u_1 + \cdots + \mu_\ell u_\ell = -(\nu_1 w_1 + \cdots + \nu_k w_k)$$

this vector is in $U_2$

Since the equality holds, both sides have to lie in $U_1 \cap U_2$

Then we get:

$$\partial_1 w_1 + \cdots + \partial_k w_k \in U_1 \cap U_2.$$

Then $\partial_1 w_1 + \cdots + \partial_k w_k$ has to be a linear combination of $v_1, \ldots, v_s$. But the $v$'s and the $w$'s together form a <u>basis</u> of $U_2$, so the only way a linear combination of $w_1, \ldots, w_k$ could equal a linear combination of $v_1, \ldots, v_\ell$ is if all the coefficients are zero.

Thus $\partial_1 = \cdots = \partial_k = 0$.

Similarly, by using linear independence of $\{v_1, \ldots, v_s, u_1, \ldots, u_\ell\}$ in $U_1$, we get that all the $\mu$'s are $0$.

But then $\lambda_1 v_1 + \cdots + \lambda_\ell v_\ell = 0$, so all the $\lambda$'s are $0$ by the linear independence of $\{v_1, \ldots, v_\ell\}$, and we are done.