Recall: A basis of a vector space \( V \) is:
- a maximal linearly independent set of vectors, which is equivalent to being
- a minimal spanning set.

We say that \( V \) is finite-dimensional if it has a finite basis.

Today:
1) Example of an infinite-dimensional vector space
2) Direct sums, and the dimension formula.

1) An important infinite-dim. space over \( \mathbb{R} \):

Consider the space of all functions \( f : \mathbb{R} \to \mathbb{R} \).
We can put various conditions:
- continuous, or
- smooth, or
- integrable.

All these spaces are still infinite-dimensional.

To prove this, all we have to do is make an example of an infinite linearly independent system of functions in our space.

For example, for continuous functions,

\[
\begin{cases}
0 & \text{on } \left[ \frac{1}{n+1}, \frac{1}{n} \right] \\
\rightarrow & \text{make a function } \neq 0 \text{ on } \left[ \frac{1}{n+1}, \frac{1}{n} \right] \\
0 & \text{elsewhere.}
\end{cases}
\]

(you could also smoothen this to make it look like this: \( \frac{1}{\sqrt{3}} \))
Why are such functions linearly independent? First, what does it mean for an infinite set of vectors to be linearly independent? We say a collection $W \subseteq V$ is linearly independent if $\forall w_1, \ldots, w_n \in W, \lambda_1, \ldots, \lambda_n \in \mathbb{F}, \lambda_1 w_1 + \cdots + \lambda_n w_n = 0$ implies that $\lambda_1 = \cdots = \lambda_n = 0$ (i.e., no finite linear combination of vectors in $W$ is allowed to be $0$ unless all the coefficients are $0$).

The main example of a linearly independent infinite set is the space of all functions $f: \mathbb{R} \to \mathbb{R}$: the delta functions.

Define, for every $a \in \mathbb{R}$,

$$\delta_a(x) = \begin{cases} 1 & x = a \\ 0 & x \neq a \end{cases}$$

The collection $\{\delta_a(x) \mid a \in \mathbb{R}\}$ is linearly independent (will check in a moment). The above example is a "smoothing."
of this example.

Why are the delta-functions linearly independent:

suppose \( \lambda_1 \delta_{a_1} + \ldots + \lambda_n \delta_{a_n} = 0 \)

\[ \text{zero function} \]

i.e. it equals 0 at every \( x \in \mathbb{R} \)

Then, in particular, at \( x = a_i \), we have

\[ \lambda_1 \delta_{a_i} (a_i) + \ldots + \lambda_n \delta_{a_i} (a_i) = 0 \]

\[ \lambda_1 \delta_{a_i} (a_i) \]

\[ 0 + \ldots + \lambda_1 1 + \ldots + 0 = 0 \]

Thus, \( \lambda_i = 0 \) for all \( i \).

Next: sums and direct sums of linear subspaces.
Def: Let $U_1, U_2$ be linear subspaces of $V$.
$U_1 + U_2 = \{ x + y \mid x \in U_1, y \in U_2 \}$.

Exer: Show that $U_1 + U_2$ is a linear subspace of $V$.

Examples:
1) $u_1, u_2 \subseteq \mathbb{R}^2$

2) in $\mathbb{R}^3$:

Prop: Let $U_1, U_2$ be linear subspaces of $V$.
assume $U_1, U_2$ are finite-dimensional.

Then $\dim(U_1 + U_2) + \dim(U_1 \cap U_2) = \dim(U_1) + \dim(U_2)$

Check: in our example in $\mathbb{R}^3$:
$\dim(U_1) = \dim(U_2) = 2$
$\dim(U_1 \cap U_2) = 1$
$\dim(U_1 + U_2) = 3$

Question: in $\mathbb{R}^4$, do two planes containing 0 have to have a common line?

Yes: $\mathbb{R}^4 = L(e_1, \ldots, e_4)$. Take $U_1 = L(e_1, e_2)$
$U_2 = L(e_2, e_3)$
Def: If $U_1 \cap U_2 = \{0\}$ and $U_1 + U_2 = V$ then we say that $V$ is a 'direct sum' of $U_1$ and $U_2$, write $V = U_1 \oplus U_2$.

In our $\mathbb{R}^4$ example, check:

$\mathbb{R}^4 = \text{L}(e_1, e_2) \oplus \text{L}(e_3, e_4)$

Proof of the dimension formula:

Consider $U_1 \cap U_2$; let $v_1, \ldots, v_s$ be its basis.

Use basis extension to extend it to $\{v_1, \ldots, v_s, u_1, \ldots, u_t\}$ - basis of $U_1$

$\{v_1, \ldots, v_s, w_1, \ldots, w_k\}$ - basis of $U_2$

Claim: $\{v_1, \ldots, v_s, u_1, \ldots, u_t, w_1, \ldots, w_k\}$ is a basis of $U_1 + U_2$.

The formula follows:

$\dim(U_1 + U_2) = s + t + k$

$\dim(U_1) + \dim(U_2) = (s+t) + (s+k)$.

Proof of the claim: Why this set spans $U_1 + U_2$? 

Exercise.
Why is this set linearly independent?

Suppose we had some linear combination

$$\lambda_1 u_1 + \lambda_2 u_2 + \mu_1 u_1 + \ldots + \mu_n u_n = 0$$

(I am using different letters $\lambda, \mu$ for scalars to make it easier to keep track of which vectors they come with).

We need to prove that all $\lambda$'s, $\mu$'s and $\nu$'s are 0.

We have:

$$\sum \lambda_i u_i + \sum \mu_i u_i = 0$$

Since the equality holds, both sides have to lie in $U_1 \cap U_2$.
Then we get:
\[ \lambda_1 w_1 + \cdots + \lambda_k w_k \in U_1 \cap U_2. \]
Then \( \lambda_1 w_1 + \cdots + \lambda_k w_k \) has to be a linear combination of \( v_1, \ldots, v_s \). But the \( v_i \)'s and the \( w_i \)'s together form a basis of \( U_2 \), so the only way a linear combination of \( w_1, \ldots, w_k \) could equal a linear combination of \( v_1, \ldots, v_s \) is if all the coefficients are zero.
Thus, \( \lambda_1 = \cdots = \lambda_k = 0 \).

Similarly, by using linear independence of \( \{v_1, \ldots, v_s, u_1, \ldots, u_e\} \) in \( U_1 \), we get that all the \( \mu_i \)'s are 0.

But then \( \lambda_1 u_1 + \cdots + \lambda_k u_k + \mu_1 v_1 + \cdots + \mu_e v_e = 0 \), so all the \( \lambda_i \)'s are 0 by the linear independence of \( \{u_1, \ldots, u_e\} \), and we are done.