Some more problems for Math 322.

Note: three of these problems come from this year's Math 422. They can be found at `http://www.math.ubc.ca/~lior/`

1. Groups

1. Problem 1 from Math 422, Problem Set 1.
2. Problem 2 from Math 422, Problem Set 1.
3. (a) Let G be a finite abelian group of order n for which the number of solutions of $x^m = e$ is at most m for any m dividing n. Prove that G must be cyclic. (Hint: use that $\sum_{m|n} \varphi(m) = n$).
   (b) Let $q = p^r$ for some prime number p and some $r \geq 1$. Prove that the group $\mathbb{F}_q^*$ is cyclic.
4. (a) Let G be a group of order p^2, where p is a prime. Prove that G has a normal subgroup of order p.
   (b) Prove that any group of order p^2, where p is a prime, must be abelian.

2. Rings

5. Prove Gauss' lemma: If a monic polynomial $f \in \mathbb{Z}[x]$ is irreducible as an element of the ring $\mathbb{Z}[x]$, then it is also irreducible as an element of $\mathbb{Q}[x]$.
6. Prove Eisenstein's criterion for irreducibility of polynomials: Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots = a_0$ be a polynomial with integer coefficients. Suppose that there exists a prime number p such that p divides a_i for all $i = 0, \ldots, n-1$, and p^2 does not divide a_0. Prove that f is irreducible in $\mathbb{Q}[x]$.
7. Let $\omega = e^{2\pi i/3}$. Prove that $\mathbb{Z}[\omega]$ is a Euclidean ring.

8**. Prove that $\mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$ is a PID.

9. Problem 5 from Math 422, Problem Set 1.