Math 323. Midterm Exam. February 28, 2013. Time: 75 minutes.

(1) Let $f \in \mathbb{Z}[x]$ be a monic polynomial, and suppose $f(\alpha) = 0$ for some $\alpha \in \mathbb{Q}$. Prove that then $\alpha \in \mathbb{Z}$.

(2) (a) Is 13 prime in $\mathbb{Z}[i]$? (If not, factor it into primes).
(b) Prove that any ideal in $\mathbb{Z}[i]$ contains a positive integer.
(c) Let (3) be the ideal generated by the element 3 in $\mathbb{Z}[i]$. Describe the quotient $\mathbb{Z}[i]/(3)$.

(3) Construct a field of 27 elements.

(4) Let $R = \mathbb{Z}[\frac{\sqrt{-3}+1}{2}]$. Is it true that in the polynomial ring $R[x]$ the prime elements are the same as irreducible elements?

(5) Find an example of an element of $\mathbb{Z}[\sqrt{-5}]$ that is irreducible but not prime. (and give a complete proof that it has this property).

(6) Is the polynomial $x^6 + 30x^5 - 15x^3 + 6x - 120$ irreducible in $\mathbb{Q}[x]$?