Math 323. Practice problems on modules.

(1) Let R be an integral domain, and let F be a free module over R of finite rank, and let R^2 be the free module of rank 2 over R. Prove that $\mathrm{Hom}_R(F, R^2) \equiv F \oplus F$.

(2) Let R be an integral domain, and let M be a torsion module over R, and F – a free module over R. Prove that $\mathrm{Hom}_R(M, F) = \{0\}$. Is this statement still true if R is not an integral domain?

(3) Give an example of three modules A, B and C over a PID such that $A/B \simeq C$, but A is not isomorphic to $B \oplus C$.

(4) Let p and q be primes, and let m be an arbitrary integer. Compute

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}).$$

(5) Let I and J be two ideals in $\mathbb{Z}[i]$. Then we can think of them as $\mathbb{Z}[i]$-modules. Find $\mathrm{Hom}_{\mathbb{Z}[i]}(I, J)$.

(6) Consider the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ which acts as the orthogonal projection onto the x-axis (in the standard basis), that is, in the standard coordinates, if $\bar{v} = \langle x, y, z\rangle$, then $T\bar{v} = \langle x, 0, 0\rangle$. Consider $\mathbb{R}^3$ as an $\mathbb{R}[x]$-module via T.
   (a) Decompose it into a direct sum of cyclic modules (using any method you like, including guessing).
   (b) Find the invariant factors of this module.
   (c) Write the rational canonical form for T, and Jordan canonical form.

(7) Consider the submodule N of $\mathbb{Z}^2$ generated by the vectors $\langle 2, 4\rangle$ and $\langle 4, 10\rangle$. Describe the quotient module $\mathbb{Z}^2/N$.

(8) Prove that any module over $\mathbb{Z}[i]$ is a direct sum of a free module and a torsion module. Is the same true for modules over $\mathbb{Z}[\sqrt{-5}]$?

(9) (This requires Jordan canonical form). Classify all nilpotent linear operators in a 4-dimensional vector space over $\mathbb{C}$, up to conjugacy.

(10) (a) Make an example of a linear operator on some vector space V over $\mathbb{R}$, such that the corresponding $\mathbb{R}[x]$-module has invariant factors $x - 2$, $(x - 2)^2$, $(x - 2)^3(x - 3)$.
   (b) For this linear operator, write down its characteristic and minimal polynomials.
   (c) Write the Jordan canonical form for the matrix from (a).

(11) (a) Prove that if two 3×3-matrices have the same characteristic polynomial and the same minimal polynomial, then they are similar.
   (b) Make an example of two 4×4-matrices that have the same characteristic and minimal polynomials, but are not similar.