Problem Set 10 (and last) – all these problems are for extra credit; you do not have to turn it in. Due Tuesday April 8; no extensions: solutions will be posted right away.

1. Section 10.3, Problem 17.

2. Section 10.3, Problem 18.

Hint: We have proved in class, using Chinese Remainder Theorem for modules (Problem 17 in 10.3), that in the situation of this problem,

$$M \simeq M/(p_1^{\alpha_1})M \times \cdots \times M/(p_n^{\alpha_n})M.$$

It remains to prove that $M_{p_i} \simeq M/(p_i^{\alpha_i})M$. To do that, consider the annihilator of the ideal $(p_i^{\alpha_i})$ in $M/(p_1^{\alpha_1})M \times \cdots \times M/(p_n^{\alpha_n})M$, and prove that it is exactly $M/(p_i^{\alpha_i})M$.

3. (a) Prove that two vectors $\bar{v}_1$, $\bar{v}_2$ form a basis of the module $\mathbb{Z}^2$ if and only if the parallelogram spanned by them contains no lattice points. Suppose that $\bar{v}_1 = \langle a_1, b_1\rangle$ and $\bar{v}_2 = \langle a_2, b_2\rangle$. Let $A = \left[\begin{smallmatrix} a_1 & b_1 \\ a_2 & b_2 \end{smallmatrix}\right]$. Prove that the above conditions are also equivalent to $\det(A) = 1$.

(b) Let N be a submodule of $\mathbb{Z}^n$, and let $A = (a_{ij})_{i=1}^n$ be the relations matrix for the generators of N with respect to the standard basis of $\mathbb{Z}^n$ (that is, the generators of N are:

$$y_1 = a_{11}e_1 + \dots a_{1n}e_n$$
$$\dots$$
$$y_n = a_{n1}e_1 + \dots a_{nn}e_n,$$

where $e_j = (0, \dots 0, 1, 0 \dots 0)$ with 1 in the jth place). Prove that

$$|\mathbb{Z}^n/N| = |\det(A)|.$$

4. (a) Sketch the ideal generated by $1 + 2i$ in $\mathbb{Z}[i]$. Find $|\mathbb{Z}[i]/(1 + 2i)|$.

(b) Sketch the ideal generated by 2 in $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$. Find $\left|\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]/(2)\right|$.

(c) Let $\mathcal{O}$ be a quadratic integer ring, and let $(a) \subset \mathcal{O}$ be a principal ideal in it. Recall that we have the notion of norm on quadratic integer rings. Prove that $|\mathcal{O}/(a)| = |N(a)|$, where $|N(a)|$ is the absolute value of the norm of a.

Hint: First show that $\mathcal{O}$ is a free rank 2 $\mathbb{Z}$-module. Then find a convenient set of generators of (a) as a $\mathbb{Z}$-submodule, and use the previous problem.

Suggested problems: Section 12.1, Problems 2, 3, 7, 10, 11, 12.