Problem Set 9. Due Tuesday April 1 (seriously).

1. Let R be an integral domain, and let M be an R-module. Prove that if $\mathrm{Tor}(M) \neq \{0\}$, then M cannot be free on any set of generators.

   Note that as proved in the last homework, if R has zero divisors, then any R-module has torsion elements, so it makes sense to discuss this condition only for integral domains.

   Remark. Let R be an integral domain that is not a PID, and let I be a non-principal finitely generated ideal in R. Then we proved in class that I considered as an R-module is torsion-free (i.e. $\mathrm{Tor}(I) = \{0\}$), but not free. Thus we see that over a general integral domain, being torsion-free is necessary but not sufficient for being free. We will prove soon that over a PID this condition is also sufficient.

2. Section 10.2, Problem 12.

3. Section 10.3, Problem 2.

4. Section 10.3, Problem 5.

5. Section 10.3, Problem 6.

6. Section 10.3, Problem 9.

7. Section 10.3, Problem 13. (*Hint: use the universal property.*)

8. Section 10.3, Problem 23.

Suggested problems, not to be handed in:

Section 10.3: Problems 16-18 and 22, and Problems 24 and 27. They will be to some extent discussed in lecture next week.

One more exercise about a universal property. Let A, B be arbitrary sets. Let $C = A \times B$ be their Cartesian product

$$C = \{(a, b) \mid a \in A, b \in B\},$$

and let $\pi_1 : C \to A$, $\pi_2 : C \to B$ be the projections onto the first and second factor: $\pi_1(a, b) = a$, $\pi_2(a, b) = b$. Prove that the Cartesian product $C = A \times B$ with the projections π_1 and π_2 satisfies the following universal property: for any set M and any two maps $f_1 : M \to A$ and $f_2 : M \to B$, there exists a unique map $\Phi : M \to C$, such that $f_1 = \pi_1 \circ \Phi$, $f_2 = \pi_2 \circ \Phi$. Prove that if D is any other set (equipped with maps $p_1 : D \to A$ and $p_2 : D \to B$) satisfying the same universal property, then there exists a bijective map from C to D, such that it commutes with the projections π_1, p_1 and π_2, p_2.