Some more (optional) problems for Math 323.

This is a compilation of both basic exercises and harder problems that are useful/interesting, but were not included in the homework; many of these are from Dummit and Foote. Solve as many as you like. Please do not hand in. Solutions will be posted before the midterm.

1. Divisors and units in non-commutative rings. Let R be a ring with $1 \neq 0$.

(1) Prove that u is a unit if and only if it has both a right and a left inverse.

(2) Prove that if u has a right inverse, then u is not a right zero divisor.

(3) Prove that if u has more than one right inverse then u is a left zero divisor.

(4)* Prove that if u has more than one right inverse then u has infinitely many right inverses.

(5) Prove that if R is a finite ring then every element that has a right inverse is a unit. (i.e. has a two-sided inverse)

(6) Give an example of a left zero-divisor in some ring which is not a right zero-divisor.

(7) Give an example of a left-invertible element which is not right-invertible.

(8) Let R be a ring with identity. If $1 - ab$ is a unit in R, prove that $1 - ba$ is also a unit in R.

2. Idempotents.

(1) Let R be a ring such that for all $r \in R$, if $r^2 = 0$ then $r = 0$. Prove that all idempotents are central (Recall that an element $a \in R$ is idempotent if $a^2 = a$, and "central" means elements of $Z(R)$).

(2) Let R be a ring such that $r^2 - r \in Z(R)$ for every $r \in R$. Prove that R is commutative. (This problem generalizes the statement from last homework, "If $r^2 = r$ for every $r \in R$, then R is commutative.")

(3) Let $M_2(F)$ be the ring of 2×2-matrices over a field F. Let A be an element $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Prove that $M_2(F) \simeq J_1 \times J_2$, where J_1 is the ideal $J_1 = \{XA \mid X \in M_2(F)\}$ and $J_2 = \{X(I - A) \mid X \in M_2(F)\}$, where I is the identity matrix. (Hint: see Question 2 in Homework 4). **State this result in terms of linear operators acting on a plane, and generalize it to higher dimensions.**

3. Homomorphisms.

(1) Let A be any commutative ring with identity $1 \neq 0$. Let R be the set of all group homomorphisms of the additive group A to itself with addition defined as pointwise addition of functions and multiplication defined as function composition. Prove that these operations make R into a ring with identity. Prove that the units of R are the group automorphisms of A.

(2) Prove that the rings $2\mathbb{Z}$ and $3\mathbb{Z}$ are not isomorphic.

(3) Describe all ring homomorphisms from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$. In each case, describe the kernel and the image.

4. Quadratic Integer Rings.

Let D be an integer that is not a perfect square in $\mathbb{Z}$ and let

$$S = \left\{ \begin{pmatrix} a & b \\ Db & a \end{pmatrix} : a, b \in \mathbb{Z} \right\}.$$

(1) Prove that S is a subring of $M_2(\mathbb{Z})$.

(2) If D is not a perfect square in $\mathbb{Z}$, prove that the map $\phi : \mathbb{Z}[\sqrt{D}] \to S$ defined by $\phi(a+b\sqrt{D}) = \begin{pmatrix} a & b \\ Db & a \end{pmatrix}$ is a ring isomorphism. **Note:** This isomorphism is useful for the study of the algebraic group of invertible 2×2 matrices (which is part of Lie theory).

(3) If $D \equiv 1 \mod 4$ is square-free, prove that the set $\left\{ \begin{pmatrix} a & b \\ (D-1)b/4 & a+b \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ is a subring of $M_2(\mathbb{Z})$ and is isomorphic to the quadratic integer ring $\mathcal{O}$.

5. Prime and Maximal ideals.

(1) Prove that if M is an ideal such that R/M is a field then M is a maximal ideal. (do not assume R is commutative)

(2) Let R be a commutative ring in which every proper ideal is a prime. Prove that R is a field.