1   Pell's equation and continued fractions

This is a series of (optional) problems about Pell's equation:

$$x^2 - Dy^2 = 1 \qquad (1)$$

that leads to understanding the set of its solutions, and therefore, of the units in the ring $\mathbb{Z}[\sqrt{D}]$. It does not use any sophisticated theory, but instead uses continued fractions (introduced below).

1.1   warm-up

1. Let k be an integer, and suppose the pair of integers (x, y) solves the equation $x^2 - 2y^2 = k$. Then find a pair of integers (t, u) such that $t^2 - 2u^2 = -k$.

 In some special cases, one can find a solution "by hand":

2. Find a solution to equation (1) if $D = a(a + 1)$ for some positive integer a.

3. Find a solution if $D = a^2 - 1$.

4. Find a solution if $D = a^2 + 1$ for some positive integer a.

1.2   The general case

Everywhere below by a "solution" we will mean a pair of *positive integers* (x, y) that satisfy the equation (1).

5. Show that if (x, y) and (x_1, y_1) are solutions, and the pair of integers (a, b) satisfies $(x + y\sqrt{D})(x_1 + y_1\sqrt{D}) = a + b\sqrt{D}$, then (a, b) is a solution as well. [1]

6. Show that if (x, y) is a solution then $x/y - \sqrt{D} < 1/y^2$.

 The last problem suggests that we should look for solutions among numerators and denominators of rational numbers that are close to $\sqrt{D}$.

 Suppose we are given an irrational number x. Then we can write its decimal expansion, and by cutting it off after n digits we can get rational approximations, call them x_n, to the number x. However, these approximations are, in some sense, not the best possible. Indeed, suppose $x_n = p_n/q_n$ (thus, q_n is a power of 10). Then what can we say about $|x - x_n|$? It is approximately of the order of magnitude of $1/q_n$ (Why?). Suppose we want to do much better – we want rational approximations p_n/q_n such that $|x - p_n/q_n|$ is of the same order of magnitude as $1/q_n^2$. Such an approximation can be achieved by taking a *continued fraction* expression for x, and cutting it off after n "digits". (in fact,

---

[1] In other words, the product of units in the ring of integers of $\mathbb{Z}[\sqrt{D}]$ is a unit.

one can prove that the continued fraction for x gives the best possible sequence of rational approximations for x, in general).

A continued fraction is an expression of the form

$$q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + ...\cfrac{1}{q_n + ...}}}}.$$

As usual, one says that a number x is represented by this continued fraction if the sequence of numbers one gets by truncating it at q_n converges to x.

7. Show that if the sequence of coefficients $\{q_0, \ldots, q_n, \ldots\}$ is periodic, then the number represented by this continued fraction is a solution to a quadratic equation with rational coefficients.

In fact, the converse is also true.

In this sense the roots of quadratic equations with rational coefficients play the same role for continued fraction expansions as the rationals play for decimal expansions.

You can use the following fact without proof in all the remaining problems (or prove it).

The continued fraction for $\sqrt{N}$ has the form

$$q_0, \overline{q_1, q_2, ..., q_{n-1}, q_n, 2q_0}.$$

This notation means that q_0 is the integral part, and the sequence of coefficients $\{q_n\}$ is periodic, with the period being all the numbers listed under the line.

It also turns out that for a continued fraction representing $\sqrt{N}$ the sequence $\{q_1, \ldots, q_n\}$ is palindromic.

Now we can describe the smallest positive integral solution to Pell's equation; then we can prove that all solutions can be obtained from it.

Consider the two fractions we get just before the sequence $\{q_n\}$ starts repeating itself, namely, let

$$\frac{A_n}{B_n} = q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + ...\cfrac{1}{q_n}}}}.$$

$$\frac{A_{n-1}}{B_{n-1}} = q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + ...\cfrac{1}{q_{n-1}}}}};$$

2

$$\frac{A_n}{B_n} = q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + ...\cfrac{1}{q_n}}}}.$$

We have:

$$\sqrt{D} = \frac{\alpha_{n+1}A_n + A_{n-1}}{\alpha_{n+1}B_n + B_{n-1}},$$

where

$$\alpha_{n+1} = 2q_0 + \cfrac{1}{q_1+}... = \sqrt{N} + q_0.$$

8. Show that $x = A_n$ and $y = B_n$ form a solution to the equation $x^2 - Dy^2 = (-1)^{n-1}$. This gives a solution to (1) when n is odd.

9. When n is even, we get a solution from the next repetition: (A_{2n}, B_{2n}) is a solution.

10. Prove that all the solutions can be obtained from this "fundamental solution" by the process from Problem 5.