Math 502. Take-home final exam. Due Tuesday April 26.
Principal series for $SL_2(\mathbb{F}_p)$.[1]
Solutions to any 5 of these problems will be sufficient to get full credit.

The goal of this problem set is to explore the two irreducible components of the representation $(\pi_{\mathrm{sgn}}, V) = \mathrm{Ind}_B^G(\mathrm{sgn})$. Let $G = \mathrm{SL}_2(\mathbb{F}_p)$, let B be the standard Borel subgroup consisting of upper-triangular matrices, and let χ be the character of B obtained from the unique order two character of $\mathbb{F}_p^\times$:

$$\chi\left(\left[\begin{smallmatrix} a & x \\ 0 & a^{-1}\end{smallmatrix}\right]\right) = \begin{cases} 1 & \text{if } a \text{ is a square in} \mathbb{F}_p^\times, \\ -1 & \text{otherwise.}\end{cases}$$

Assume that p is an odd prime. Let

$$w = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right].$$

Let N be the subgroup of unipotent upper-triangular matrices. We will need Bruhat decomposition $G = B \coprod BwN$.

(1) Prove that π_{sgn} is a direct sum of two irreducible representations. We will denote them by W_+ and W_-.

(2) Show that π_{sgn} is the restriction to $\mathrm{SL}_2(\mathbb{F}_p)$ of an irreducible representation of $\mathrm{GL}_2(\mathbb{F}_p)$. Use this fact to show that the representations W_+ and W_- have equal dimensions (equal to $\frac{p+1}{2}$). *A possibly unnecessary hint: use the proof of Proposition 24 in Section 8.1.*

(3) Recall that the vector space V of the representation π_{sgn} can be thought of as a space of complex-valued functions on G (with a certain property), on which G acts by right translations. Let

$$\varphi_1(g) = \begin{cases} \chi(g) & \text{if} \quad g \in B \\ 0 & \text{otherwise.}\end{cases}$$

Let

$$\varphi_w(g) = \begin{cases} 0 & \text{if } g \in B \\ \chi(b) & \text{if } g = bwn, \text{ with } b \in B, n \in N.\end{cases}$$

Given an additive character $\psi : \mathbb{F}_p \to \mathbb{C}$, we denote by ψ_N the corresponding character of N:

$$\psi_N\left(\left[\begin{smallmatrix} 1 & x \\ 0 & 1\end{smallmatrix}\right]\right) = \psi(x).$$

For each nontrivial additive character ψ of $\mathbb{F}_p$, let

$$\varphi_\psi(g) = \begin{cases} 0 & \text{if} \quad g \in B \\ \chi(b)\psi_N(n) & \text{if } g = bwn, \text{ with } b \in B, n \in N.\end{cases}$$

Prove that these functions form a basis of the space of π_{sgn}. Note that these functions are eigenfunctions for N.

[1]Acknowledgement: most of these problems are borrowed from W. Casselman's personal notes; I thank him for sharing these notes.

(4) Let Ω_w be the linear functional on the space V defined by

$$\Omega_w : f \mapsto \frac{1}{\sqrt{q}} \sum_{x \in N} f(wx).$$

(a) Prove that Ω_w is a homomorphism of B-representations between $\mathrm{Ind}_B^G(\chi)$ and the 1-dimensional representation χ^{-1}. (This works for an arbitrary character χ, not just for the sign character).

(b) Let T be the automorphism of the representation π_{sgn} that corresponds to Ω_w from part (a) under Frobenius reciprocity, in the case that χ is the sign character. Prove that

$$Tf(g) = \sum_{n \in N} f(wng).$$

(5) Prove that (in the notation of the previous problem):

$$\begin{aligned} T\varphi_1 &= \frac{1}{\sqrt{q}} \operatorname{sgn}(-1)\varphi_w \\ T\varphi_w &= \sqrt{q}\varphi_w \\ T\varphi_\psi &= G_{\chi,\psi}\varphi_\psi, \end{aligned}$$

where $G_{\chi,\psi} = \frac{1}{\sqrt{q}} \sum_{x \neq 0} \chi(x)\psi(x)$ is the Gauss sum.

(6) Recall from Homework 4 that $G_{\chi,\psi}G_{\chi^{-1},\psi} = \chi(-1)$. Show that $T^2 = \operatorname{sgn}(-1)I$, and that W_+ and W_- are the eigenspaces of T corresponding to the eigenvalues $\sqrt{\operatorname{sgn}(-1)}$ and $-\sqrt{\operatorname{sgn}(-1)}$, respectively.

(7) Compute the characters of the representations W_+ and W_- (let us denote these characters by ρ_+ and ρ_-); we have already proved that $\rho_+(I) = \rho_-(I) = \frac{p+1}{2}$ (where I is the identity).

(a) Compute $\rho_\pm(-I)$.

(b) Compute $\rho_\pm\left(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\right)$.

(c) Compute the values on the four non-semi-simple conjugacy classes (*Hint: the basis from Problem 3 may be useful for this).*

(d) *Prove that on the elliptic conjugacy classes the value of $\rho_\pm$ is zero. (Hint: you can use the information about the cardinalities of the conjugacy classes from the table).*