Math 502. Problem Set 1. Due Tuesday January 18, 2011

(1) Classify all degree 1 representations of the group $\mathbb{Z}/n\mathbb{Z}$, up to isomorphism.

(2) (a) Give an example of a degree 3 representation of the group A_4 of even permutations.
(b) Is this representation irreducible?

(3) Find explicitly an isomorphism between the right regular and the left regular representation of an arbitrary finite group G.

(4) Let $G = \mathbb{Z}/4\mathbb{Z}$, and let us denote its generator by 1. Let V be a complex 2-dimensional vector space, and suppose we fixed a basis e_1, e_2. Let ρ be the representation of G in V that maps 1 to the matrix $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. This representation decomposes as a direct sum of two degree 1 representations. Find them (in terms of the basis vectors e_1 and e_2). Could you have done the same over the field of real numbers? (note that in $\mathbb{R}^2$, this matrix is the matrix of the counter-clockwise rotation by $\pi/2$, in the standard basis).

(5) Let V be a two-dimensional vector space over the field of 2^n elements, where n is arbitrary. Let $\rho : \mathbb{Z}/2\mathbb{Z} \to \mathrm{GL}(V)$ be the representation defined by $\rho(1) = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ (where 1 is the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$). Prove that this representation has an invariant one-dimensional subspace which has no invariant complement. Such representations are called *indecomposable*.

(6) (ex. 2.2 in Serre) Let X be a finite set on which G acts, and let ρ be the corresponding permutation representation, and let χ_X be its character. Show that for an element $g \in G$, $\chi(g)$ is the number of elements of X fixed by g.